Curves and Surfaces

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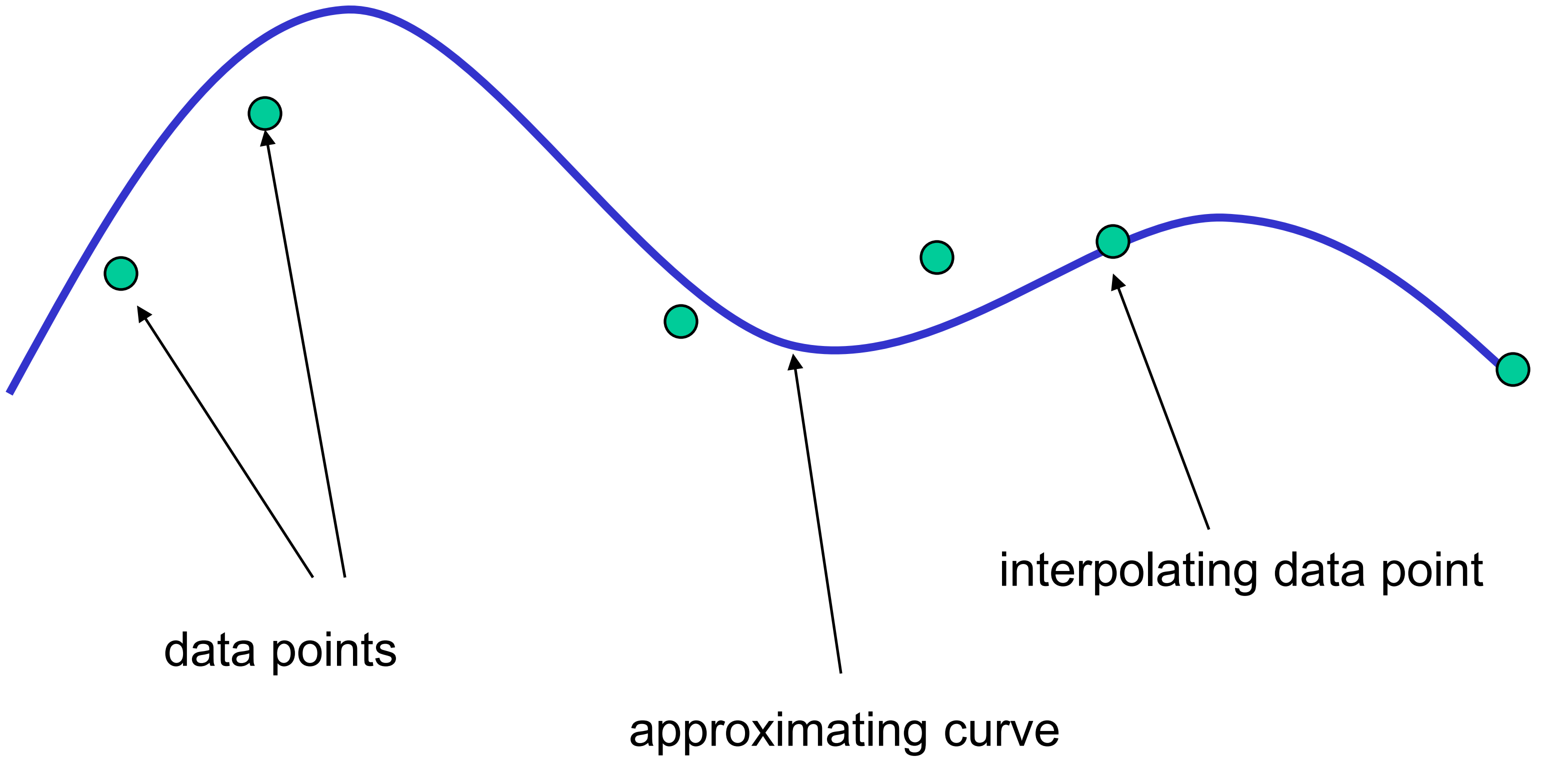
Objectives

- Introduce types of curves and surfaces
  - Explicit
  - Implicit
  - Parametric
  - Strengths and weaknesses
- Discuss Modeling and Approximations
  - Conditions
  - Stability

Escaping Flatland

- Until now we have worked with flat entities such as lines and flat polygons
  - Fit well with graphics hardware
  - Mathematically simple
- But the world is not composed of flat entities
  - Need curves and curved surfaces
  - May only have need at the application level
  - Implementation can render them approximately with flat primitives

Modeling with Curves



What Makes a Good Representation?

- There are many ways to represent curves and surfaces
- Want a representation that is
  - Stable
  - Smooth
  - Easy to evaluate
  - Must we interpolate or can we just come close to data?
  - Do we need derivatives?

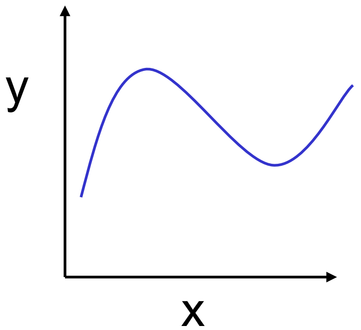
Representation of Curves & Surfaces

- Three types of object representation:
  - explicit: $y = f(x).$
  - implicit: $f(x, y) = 0.$
  - parametric: $\mathbf{p}(u) = [x(u) \;\; y(u) \;\; z(u)]^T.$

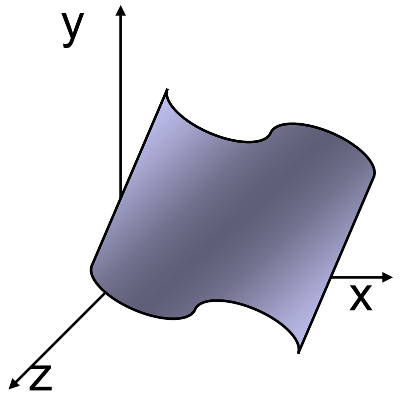
Explicit Representation

- Most familiar form of curve in 2D

$$y=f(x)$$



- Cannot represent all curves
  - Vertical lines
  - Circles
- Extension to 3D
  - $y=f(x)$, $z=g(x)$
  - The form $z = f(x,y)$ defines a surface



Explicit Representation of Lines

- The **explicit form** of a curve in 2D gives the value of one **dependent variable** in terms of the other **independent variable**.

$$y = f(x).$$

  - An explicit form may or may not exist. We write

$$y = mx + b$$

  for the line even though the equation does not hold for vertical lines.

Explicit Representation of Circles

- A circle has constant **curvature**.
  - An explicit form exists only for half of the curve:

$$y = \sqrt{r^2 - x^2}.$$

  - The other half requires a second equation:

$$y = -\sqrt{r^2 - x^2}.$$

  - In addition, we must restrict the range of *x*.
    - *f* is a function, so there must be exactly one value of *y* for every *x*.

Explicit Surfaces

- A surface requires two independent variables and two equations:

$$y = ax + b,$$

$$z = cx + d.$$

  - The line cannot be in a plane of constant *x*.
  - We cannot represent a sphere with only one equation of the form

$$z = f(x, y).$$

Implicit Representation

- An **implicit curve** has the form

$$f(x, y) = 0.$$

- Much more robust
  - A line: $ax + by + c = 0$.
  - A circle: $x^2 + y^2 - r^2 = 0$.
- Implicit functions test **membership**.
  - Does the point (x, y) lie on the curve determined by f?
- In general, there is no analytic way to find the y value for a given x.

Implicit Surfaces

- In three dimensions, a surface is described by the implicit form

$$f(x, y, z) = 0.$$

  - A plane: $ax + by + cz + d = 0$.
  - A sphere: $x^2 + y^2 + z^2 - r^2 = 0$.
- Intersect two 3D surfaces to get a 3D curve.
- Implicit curve representations are difficult to use in 3D.

Algebraic Surfaces

- One class of useful implicit surfaces is the **quadric** surface.
  - **Algebraic surfaces** are those for which the function $f(x, y, z)$ is the sum of polynomials.

$$\sum_i \sum_j \sum_k x^i y^j z^k = 0$$

  - Quadric surfaces contain polynomials that have degree at most two: $2 \geq i+j+k$

    This yields at most 10 terms

Parametric Form

- Expresses the value of each spatial component in terms of an independent variable *u*, the **parameter**:

$$x = X(u), \qquad y = Y(u), \qquad z = Z(u).$$

  - 3 explicit functions, 1 independent variable.
  - Same form in 2D and 3D.
- The most flexible and robust form for computer graphics.

Parametric Form of Line

- Parametric form of the line:
  - More robust and general than other forms
  - Extends to curves and surfaces
- Two-dimensional forms
  - Explicit: $y = mx + b$
  - Implicit: $ax + by + c = 0$
  - Parametric:

$$x(\alpha) = (1-\alpha)x_0 + \alpha x_1$$
$$y(\alpha) = (1-\alpha)y_0 + \alpha y_1$$

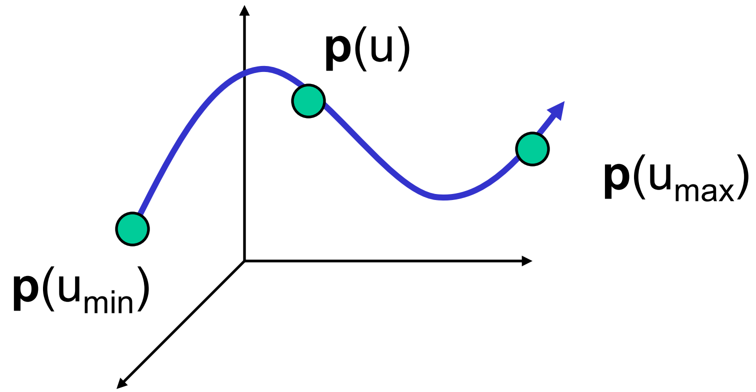
Parametric Curves

- Separate equation for each spatial variable

$$x=x(u)$$
$$y=y(u)$$
$$z=z(u)$$

Matrix notation:
$\mathbf{p}(u)=[x(u), y(u), z(u)]^T$

- The parametric form describes the locus of points being drawn as u varies: $u_{min} \leq u \leq u_{max}$



Derivative of the Curve

- The derivative is the velocity with which the curve is traced out:

$$\frac{d\mathbf{p}(u)}{du} = \begin{bmatrix} dx(u)/du \\ dy(u)/du \\ dz(u)/du \end{bmatrix}.$$

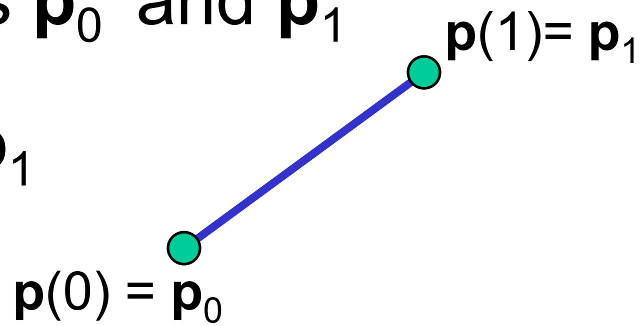
- It points in the direction tangent to the curve.

Parametric Lines

We can normalize u to be over the interval (0,1)

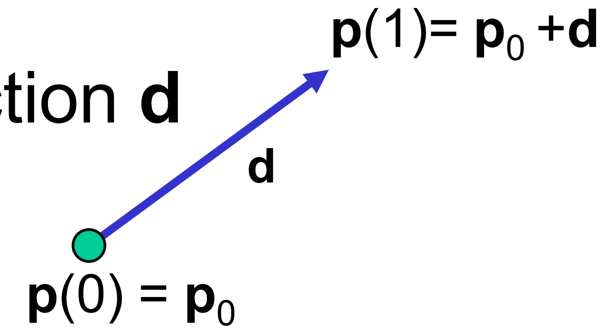
Line connecting two points $\mathbf{p}_0$ and $\mathbf{p}_1$

$$\mathbf{p}(u)=(1-u)\mathbf{p}_0+u\mathbf{p}_1$$



Ray from $\mathbf{p}_0$ in the direction $\mathbf{d}$

$$\mathbf{p}(u)=\mathbf{p}_0+u\mathbf{d}$$

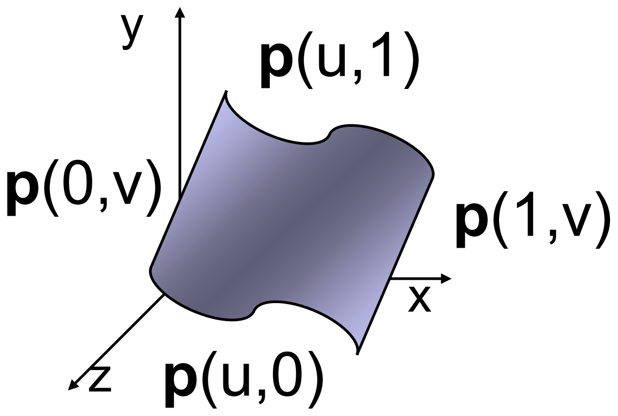


Parametric Surfaces

- Surfaces require 2 parameters

$$x=x(u,v)$$
$$y=y(u,v)$$
$$z=z(u,v)$$
$$\mathbf{p}(u,v) = [x(u,v), y(u,v), z(u,v)]^T$$

- Want same properties as curves:
  - Smoothness
  - Differentiability
  - Ease of evaluation

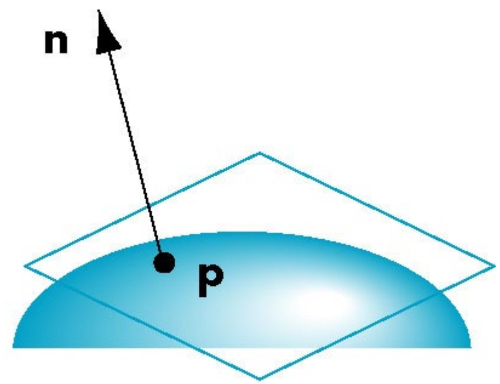


Normals

We can differentiate with respect to u and v to obtain the normal at any point **p**

$$\frac{\partial \mathbf{p}(u,v)}{\partial u} = \begin{bmatrix} \partial x(u,v)/\partial u \\ \partial y(u,v)/\partial u \\ \partial z(u,v)/\partial u \end{bmatrix}$$

$$\frac{\partial \mathbf{p}(u,v)}{\partial v} = \begin{bmatrix} \partial x(u,v)/\partial v \\ \partial y(u,v)/\partial v \\ \partial z(u,v)/\partial v \end{bmatrix}$$



$$\mathbf{n} = \frac{\partial \mathbf{p}(u,v)}{\partial u} \times \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

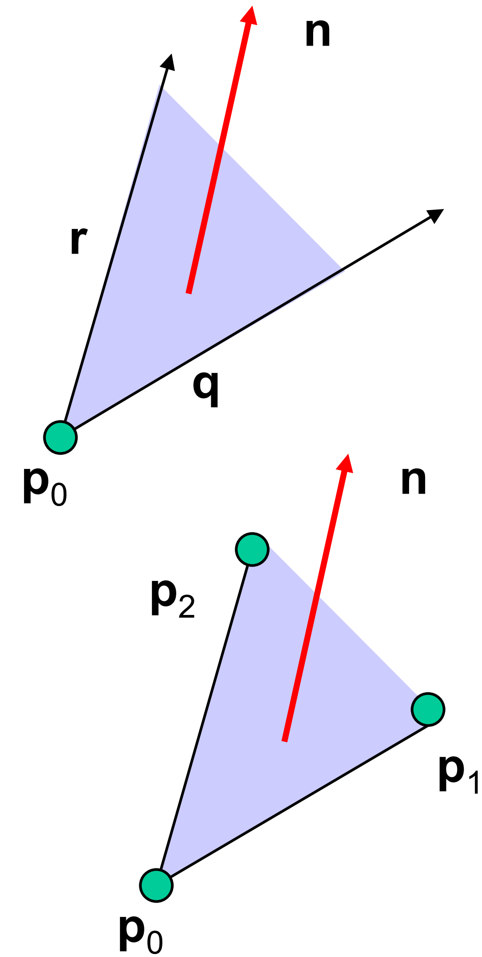
Parametric Planes

Point-Vector form

$$\mathbf{p}(u,v)=\mathbf{p}_0+u\mathbf{q}+v\mathbf{r}$$
$$\mathbf{n} = \mathbf{q} \times \mathbf{r}$$

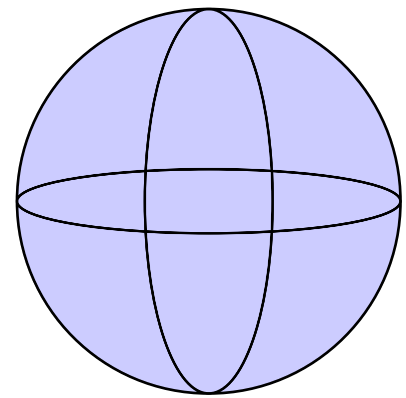
Three-point form

$$\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_0$$
$$\mathbf{r} = \mathbf{p}_2 - \mathbf{p}_0$$



Parametric Sphere

$x(u,v) = r \cos q \sin f$
$y(u,v) = r \sin q \sin f$
$z(u,v) = r \cos f$



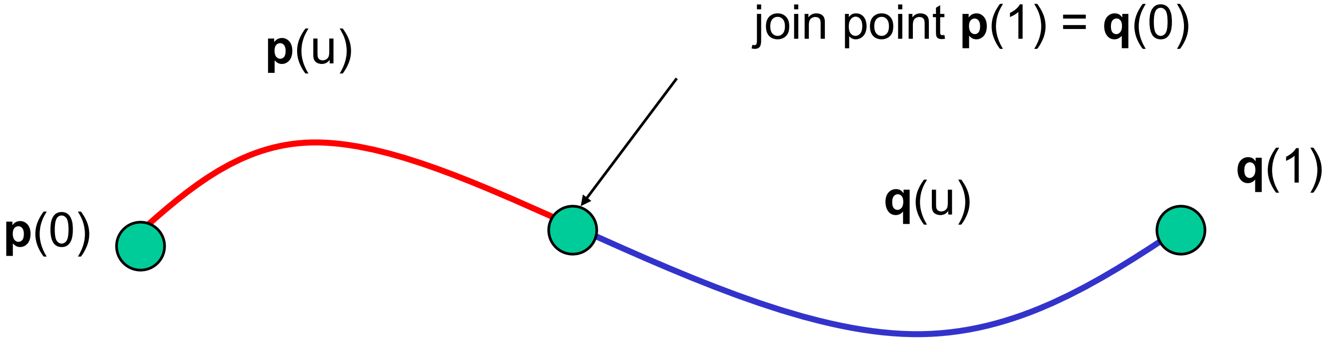
$360 \geq q \geq 0$
$180 \geq f \geq 0$

θ constant: circles of constant longitude
f constant: circles of constant latitude

differentiate to show **n** = **p**

Curve Segments

- After normalizing u, each curve is written
  $\mathbf{p}(u)=[x(u), y(u), z(u)]^T, \quad 0 \le u \le 1$
- In classical numerical methods, we design a single global curve
- In computer graphics and CAD, it is better to design small connected curve *segments*



Parametric Polynomial Curves

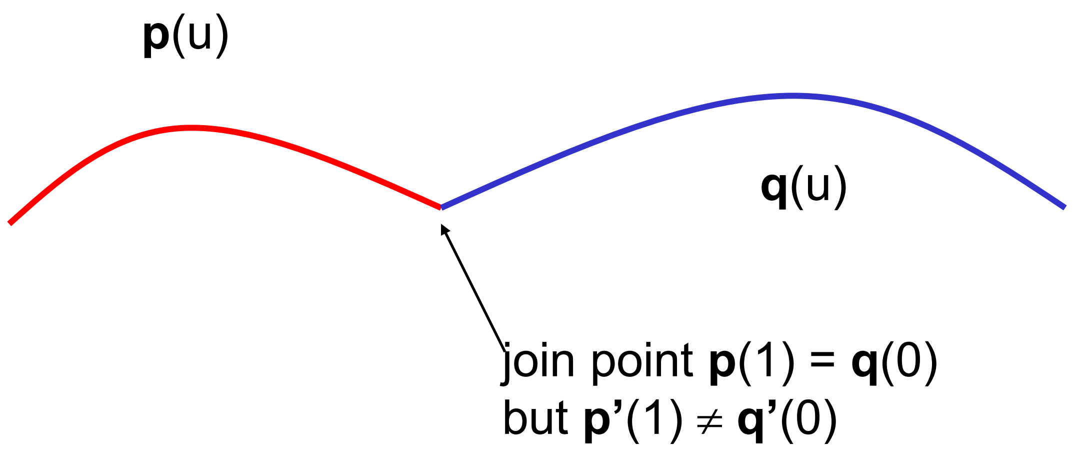
$$x(u) = \sum_{i=0}^{N} c_{xi} u^{i} \quad y(u) = \sum_{j=0}^{M} c_{yj} u^{j} \quad z(u) = \sum_{k=0}^{L} c_{zk} u^{k}$$

- If N=M=K, we need to determine 3(N+1) coefficients
- Equivalently we need 3(N+1) independent conditions
- Noting that the curves for x, y and z are independent, we can define each independently in an identical manner
- We will use the form where p can be any of x, y, z

$$\mathrm{p}(u) = \sum_{k=0}^{L} c_{k} u^{k}$$

Why Polynomials

- Easy to evaluate
- Continuous and differentiable everywhere
  - Must worry about continuity at join points including continuity of derivatives



Cubic Parametric Polynomials

- N=M=L=3, gives balance between ease of evaluation and flexibility in design

$$\mathrm{p}(u) = \sum_{k=0}^{3} c_k u^k$$

- Four coefficients to determine for each of x, y and z
- Seek four independent conditions for various values of u resulting in 4 equations in 4 unknowns for each of x, y and z
  - Conditions are a mixture of continuity requirements at the join points and conditions for fitting the data

Cubic Polynomial Surfaces

p(u,v)=[x(u,v), y(u,v), z(u,v)]^T

where

$$p(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij} u^i v^j$$

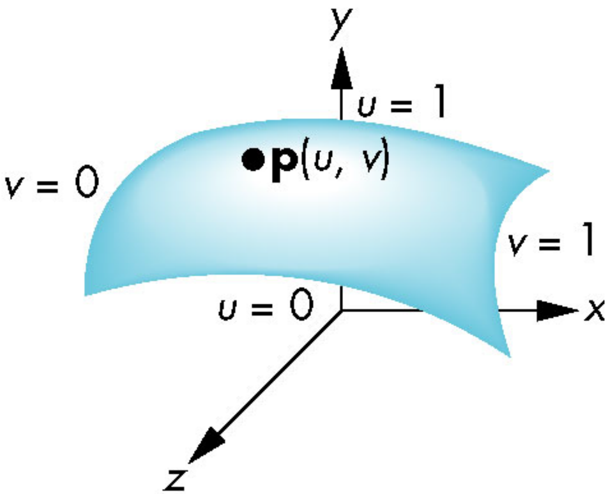
p is any of x, y or z

Need 48 coefficients (3 independent sets of 16) to determine a surface patch

Parameteric Polynomial Surfaces

In general,

$$\mathbf{p}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij} u^i v^j$$



- A **surface patch**:
  - Specify 3(n+1)(m+1) coefficients.
  - Let n = m, and let u and v vary over the rectangle $0 \le u, v \le 1$.