Geometry

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Objectives

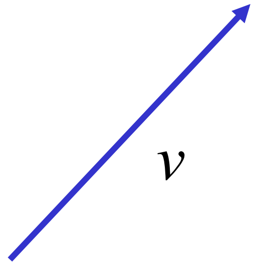
- Introduce the elements of geometry
  - Scalars
  - Vectors
  - Points
- Develop mathematical operations among them in a coordinate-free manner
- Define basic primitives
  - Line segments
  - Polygons

Basic Elements

- Geometry is the study of the relationships among objects in an n-dimensional space
  - In computer graphics, we are interested in objects that exist in three dimensions
- Want a minimum set of primitives from which we can build more sophisticated objects
- We will need three basic elements
  - Scalar: number representing magnitude
  - Vector: quantity representing magnitude and direction
  - Point: location in space

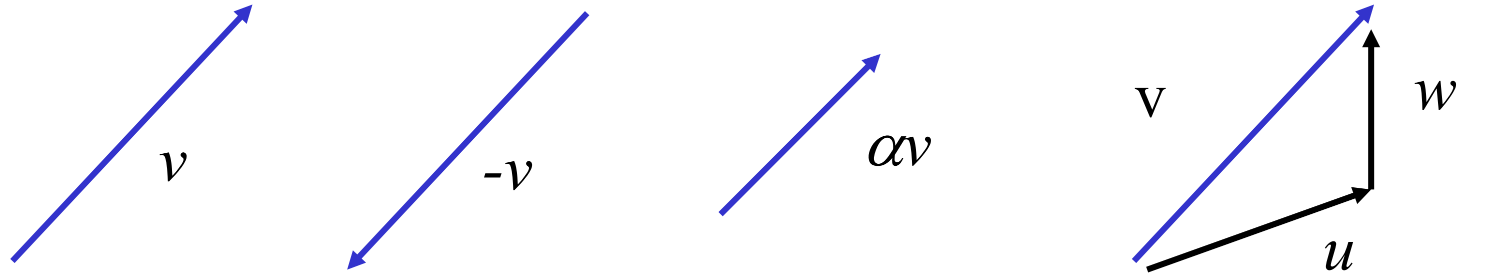
Vectors

- Physical definition: a vector is a quantity with two attributes
  - Direction
  - Magnitude
- Examples include
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types



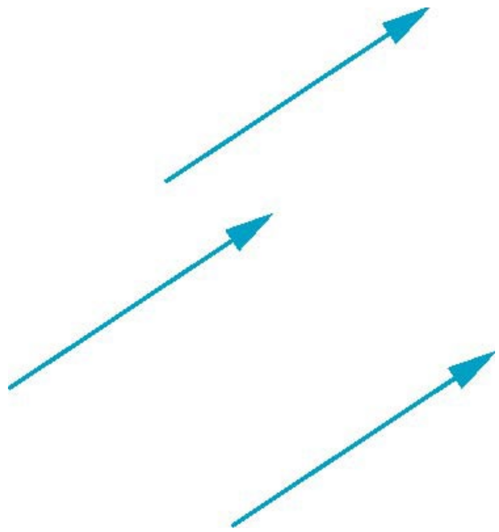
Vector Operations

- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Use head-to-tail axiom



Vectors Lack Position

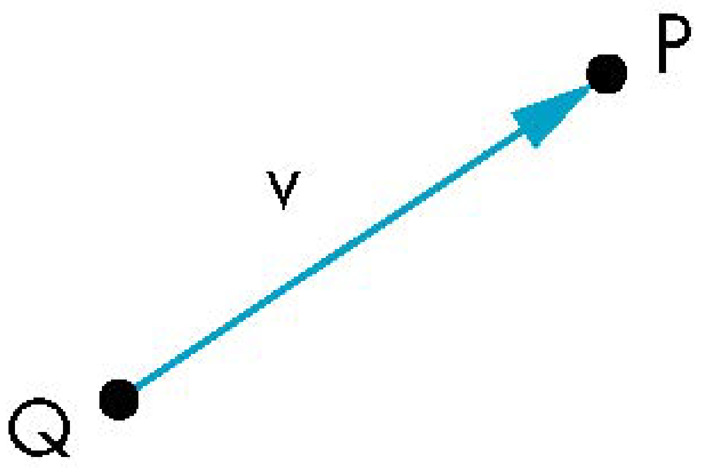
- These vectors are identical
  - Same length and magnitude



- Vectors spaces insufficient for geometry
  - Need points

Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition

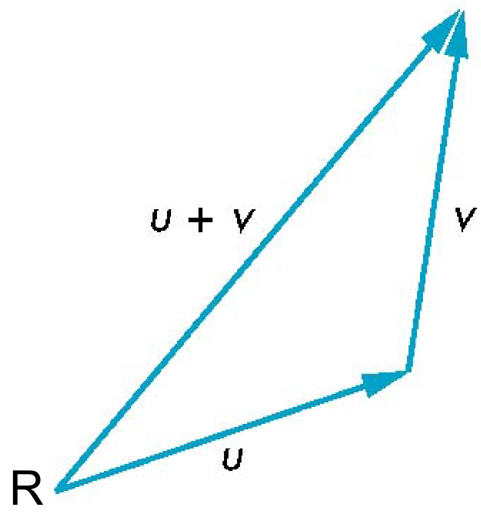


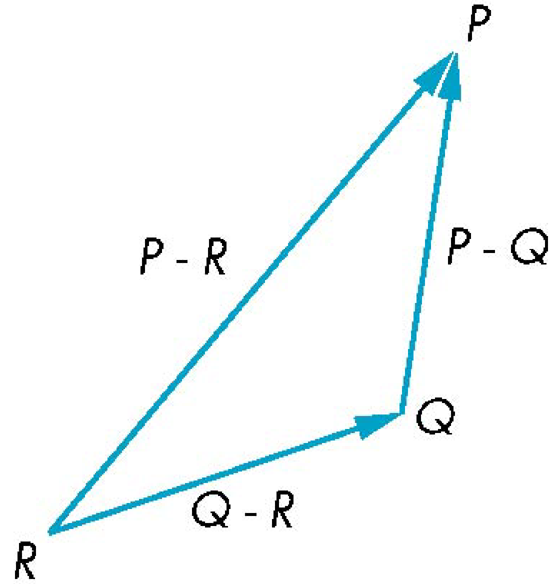
v=P-Q

P=v+Q

Planes

- A plane be determined by a point and two vectors or by three points





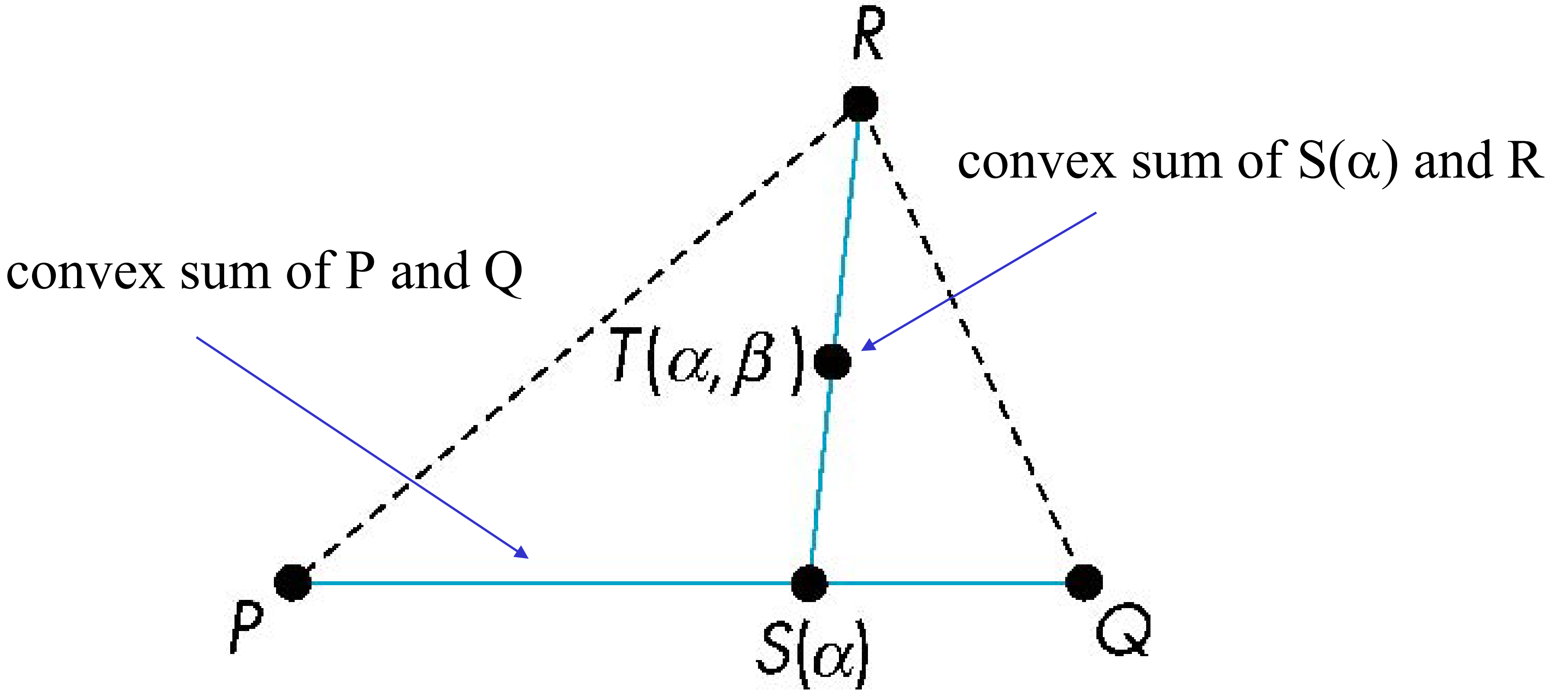
$P(\alpha,\beta)=R+\alpha u+\beta v$

(point and two vectors: R, u, v)

$P(\alpha,\beta)=R+\alpha(Q-R)+\beta(P-Q)$

(three points: R, P, Q)

Triangles



for 0<=α,β<=1, we get all points in triangle

Coordinate System

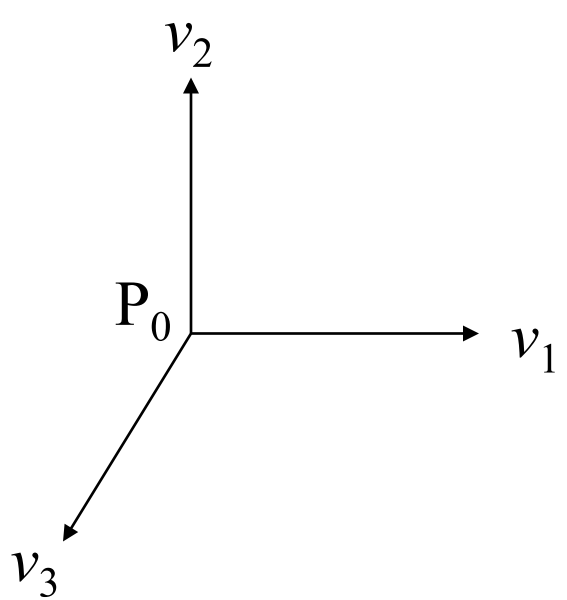
- Consider a basis $v_1, v_2, \ldots, v_n$ (such as the x, y, and z axes)
- A vector is written $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$
- The list of scalars $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is the *representation of* v *with respect to the given basis*
- We can write the representation as a row or column array of scalars: $\mathbf{a} = [\alpha_1 \ \ \alpha_2 \ \ \ldots \ \ \alpha_n]^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_n \end{bmatrix}$

Ex: $v = 2v_1 + 3v_2 - 4v_3 \longrightarrow \mathbf{a} = [2 \ 3 \ -4]^T$

This representation is with respect to a particular basis

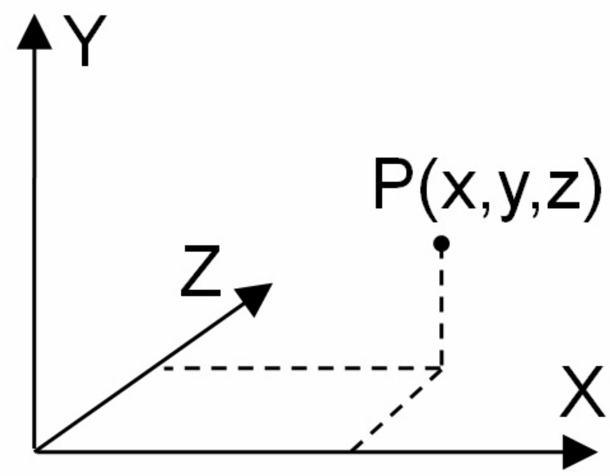
Coordinate Frame

- To form a coordinate frame, we must add a single point, the origin P_0, to the basis vectors.

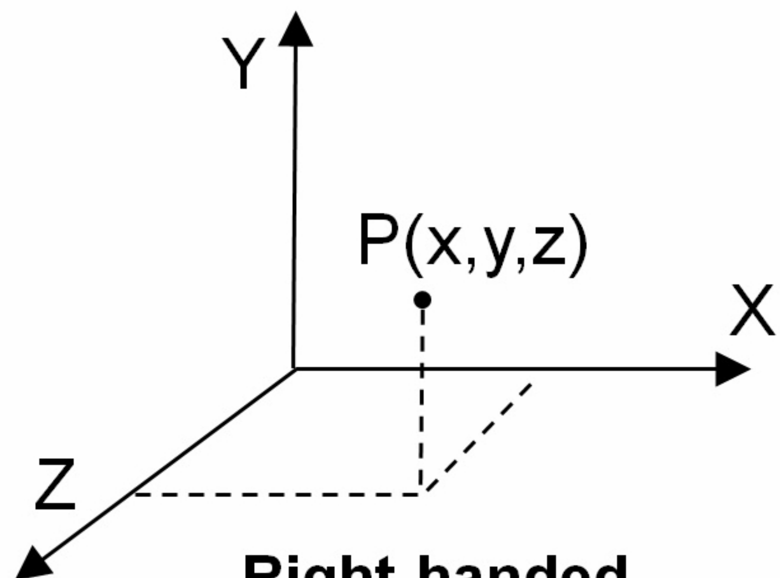


The terms coordinate system and coordinate frame are often used interchangeably.

3D Coordinate Systems



Left-handed Coordinate System



Right-handed Coordinate System

Representation of Vectors and Points

- Frame determined by (P_0, v_1, v_2, v_3)
- Within this frame, every vector can be written as

$$v=\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

Vector is just direction

- Every point can be written as

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$$

Point is anchored: displaced from origin P_0

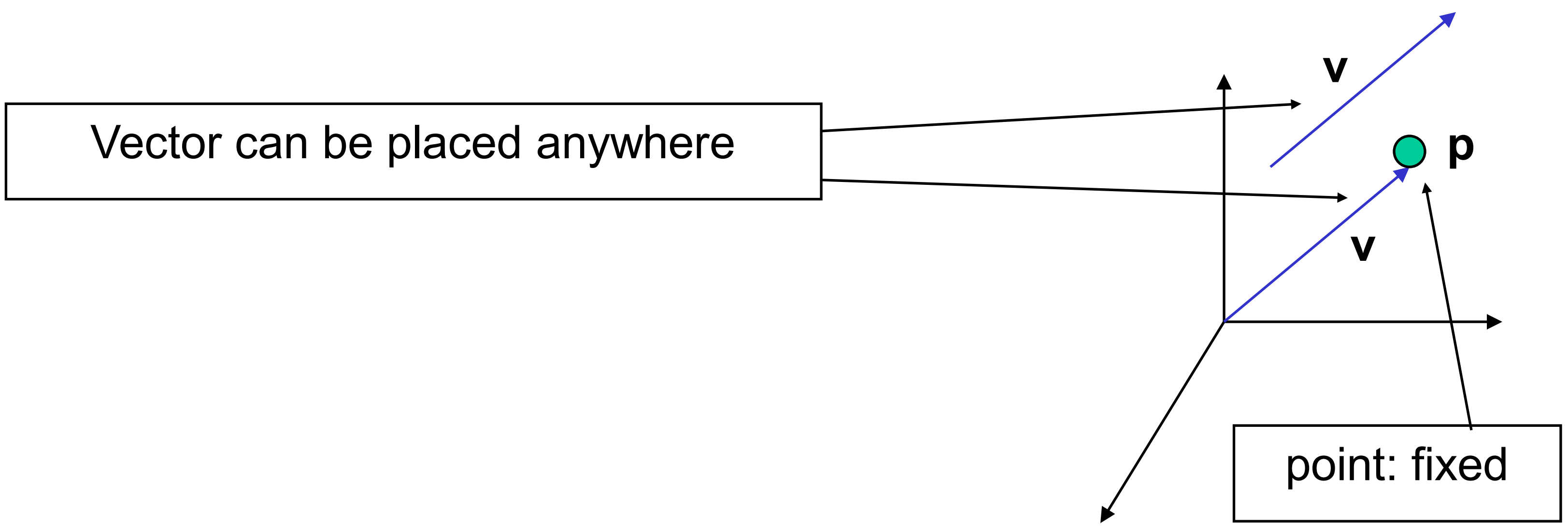
Confusing Points and Vectors

- Points and vectors appear to have similar representations

  $\mathbf{p} = [\beta_1\ \beta_2\ \beta_3]$

  $\mathbf{v} = [\alpha_1\ \alpha_2\ \alpha_3]$

- A vector has no position.



A Single Representation

If we define $0 \bullet \mathrm{P} = 0$ and $1 \bullet \mathrm{P} = \mathrm{P}$ then we can write

$$\mathrm{v} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = [\alpha_1\ \alpha_2\ \alpha_3\ 0\ ]\ [v_1\ v_2\ v_3\ \ \mathrm{P}_0]^{\mathrm{T}}$$

$$\mathrm{P} = \mathrm{P}_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = [\beta_1\ \beta_2\ \beta_3\ 1\ ]\ [v_1\ v_2\ v_3\ \ \mathrm{P}_0]^{\mathrm{T}}$$

Thus we obtain the 4D *homogeneous coordinate* representation

$$\mathbf{v} = [\alpha_1\ \alpha_2\ \alpha_3\ 0\ ]^{\mathrm{T}}$$

$$\mathbf{p} = [\beta_1\ \beta_2\ \beta_3\ 1\ ]^{\mathrm{T}}$$

3D points and vectors can be represented with a 4D homogeneous coordinate. The only difference is that a vector has a 0 as its 4th component.

Homogeneous Coordinates

The homogeneous coordinates form for a three dimensional point [x y z] is given as

$\mathbf{p} = [x' \ y' \ z' \ w]^T = [wx \ wy \ wz \ w]^T$

We return to a three dimensional point (for $w \neq 0$) by

$x \leftarrow x'/w$

$y \leftarrow y'/w$

$z \leftarrow z'/w$

If $w=0$, the representation is that of a vector

Note that homogeneous coordinates replaces points in three dimensions by lines through the origin in four dimensions

For w=1, the representation of a point is [x y z 1]

Homogeneous Coordinates and Computer Graphics

- Homogeneous coordinates are key to all computer graphics systems
  - All standard transformations (rotation, translation, scaling) can be implemented by matrix multiplications with 4 x 4 matrices
  - Hardware pipeline works with 4D representations
  - For orthographic viewing, we can maintain $w = 0$ for vectors and $w = 1$ for points
  - For perspective, we need a *perspective division*