Sampling Theory

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Objectives

• In this lecture we describe sampling theory:
  - Motivation
  - Analysis in the spatial and frequency domains
  - Ideal and nonideal solutions
  - Aliasing and antialiasing
  - Example: oversampling CD players
Sampling Theory

Sampling theory addresses the two following problems:
1. Are the samples of $g_s(x)$ sufficient to exactly describe $g(x)$?
2. If so, how can $g(x)$ be reconstructed from $g_s(x)$?
Motivation

• Improper consideration leads to jagged edges in magnification and Moire effects in minification.

Unfiltered image

Filtered image
Insight: Analysis in Frequency Domain

Spatial domain

\[ x \]

\[ g(x) \]

\[ s(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT_s) \]

\[ T_s \]

\[ g_s(x) = g(x)s(x) \]

Frequency domain

\[ S(f) = \sum_{n=-\infty}^{\infty} f_s \delta(f - nf_s) \]

\[ f_s = 1/T_s \]

\[ G_s(f) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s) \]

\[ G(f) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi ft} dx \]
Exploit Well-Known Properties of Fourier Transforms

1. Multiplication in spatial domain ↔ convolution in frequency domain.
2. Fourier transform of an impulse train is itself an impulse train.
3. $G_s(f)$ is the original spectrum $G(f)$ replicated in the frequency domain with period $f_s$.

$$G_s(f) = G(f) + G_{high}(f) \rightarrow \text{introduced by sampling}$$
Solution to Reconstruction

Discard replicas $G_{high}(f)$, leaving baseband $G(f)$ intact.

Two Provisions for exact reconstruction:

- The signal must be bandlimited. Otherwise, the replicas would overlap with the baseband spectrum.
- $f_s > 2f_{max}$ (Nyquist frequency)
Ideal Filter in the Frequency Domain: Box Filter

- Assuming provisions hold, we can discard $G_{\text{high}}$ by multiplying $G_s(f)$ with a box filter in the frequency domain.

$$H(f) = \begin{cases} 
1 & |f| < f_{\text{max}} \text{ (passband)} \\
0 & |f| \geq f_{\text{max}} \text{ (stopband)} 
\end{cases}$$
Ideal Filter in the Spatial Domain: Sinc Function

The ideal lowpass filter in the spatial domain is the inverse Fourier transform of the ideal box:

\[
sinc(x) = \frac{\sin(\pi x)}{\pi x}
\]
Reciprocal Relationship

- Reciprocal relationship between spatial and frequency domains:

- For interpolation, \( A \) must equal 1 (unity gain when sinc is centered at samples; pass through 0 at all other samples)

\[ W = f_{\text{max}} = 0.5 \text{ cycle / pixel for all digital images} \]

- Highest freq: on-off sequence
1 cycle is 2 pixels (black, white, black, white), therefore \( \frac{1}{2} \) cycle per pixel is \( f_{\text{max}} \).
Blurring

- To blur an image, \( W \downarrow \) and \( A \downarrow \) so that \( A/2W \) remains at 1.

\[
A_1 \quad \frac{1}{2}w_1 \\
A_2 < A_1 \quad W_2 < W_1
\]

- To interpolate among sparser samples, \( A=1 \) and \( W \downarrow \) which means that \((A/2W)\uparrow\). Since sampling decreases the amplitude of the spectrum, \((A/2W)\uparrow\) serves to restore it.

\[
g(x) = \text{sinc}(x) \ast g_s(x) \\
= \int_{-\infty}^{\infty} \text{sinc}(\lambda) g_s(x - \lambda) d\lambda
\]

\textbf{Note}: sinc requires infinite number of neighbours: impossible.

- Ideal low-pass filtering is impossible in the spatial domain. It is not impossible in the frequency domain for finite data streams.
Nonideal Reconstruction

- Possible solution to reconstruction in the spatial domain: truncated sinc.

- Alternate solution: nonideal reconstruction

- Nonideal because it attenuates the higher frequencies in the passband and doesn’t fully suppress the stopband.
Aliasing

If \( f_s < 2f_{\text{max}} \) our signal is undersampled.
If \( f_{\text{max}} \) is infinite, our signal is not bandlimited.
Either way, we have aliasing: high frequencies masquerade, or alias as, low frequencies.

Temporal aliasing: stagecoach wheels go backwards in films (sampled at 24 frames/sec).
Two approaches to antialiasing for combating aliasing:

1. Sample input at higher rate: increase $f_s$

2. Bandlimit input before sampling at low $f_s$
Antialiasing

Intuition for increasing $f_s$:
Higher sampling rates permit sloppier reconstruction filters.

Marginally acceptable $f_s$
Requires ideal lowpass filter

Pushed to limits; use sinc

Many samples: linear interpolation adequate

|Gs(f)|
\[ f_s \]

|Gs(f)|
\[ -f_s \]

high $f_s$ allows nonideal reconstruction filter.
This adequate because $H_r(f)$ doesn’t taper off until $|F|>|F_{max}|$
Nonideal Reconstruction Filter

\[
H_r(f) = \begin{cases} 
1 & 0 \leq |f| \leq f_{\text{max}} \\
\neq 0 & f_{\text{max}} < |f| < f_s - f_{\text{max}} \\
0 & |f| \geq f_s - f_{\text{max}} 
\end{cases}
\]
Nonideal Reconstruction Filter

Intuition for bandlimiting input:
Inability to change fs: e.g., restricted by display resolution.
For example, an image in perspective may require infinite pixels to capture all visual details. However, only finite # pixels available, e.g. 512 x 512

Many pixels in oblique image map to one output pixel. Must blur neighborhood before returning a single value to the output image. In general, dull signals don’t need bandlimiting, rich ones do.
Example: Oversampling CD players

\[ f_s = 44.1 \text{ kHz} \text{ (sample/second)} \]

\[ = 20 \text{ kHz} \times 2 + 4.1 \text{ kHz} = 44.1 \text{ KHz} \]

\( f_{\text{max}} \) Nyq rate cushion

No Oversampling

Oversampling

Sloppy reconstruction by Analog output filter (speaker)

Accurate digital Filter in CD Player reconstructs well

large overshoot/undershoots

less

more intermediate samples

Wolberg: Image Processing Course Notes
Adequate Sampling

Inadequate Sampling

Wolberg: Image Processing Course Notes
Antialiasing