Fourier Transform

Prof. George Wolberg
Dept. of Computer Science
City College of New York

Objectives

- This lecture reviews Fourier transforms and processing in the frequency domain.
  - Definitions
  - Fourier series
  - Fourier transform
  - Fourier analysis and synthesis
  - Discrete Fourier transform (DFT)
  - Fast Fourier transform (FFT)

Background (1)

- Fourier proved that any periodic function can be expressed as the sum of sinusoids of different frequencies, each multiplied by a different coefficient. → *Fourier series*
- Even aperiodic functions (whose area under the curve is finite) can be expressed as the integral of sinusoids multiplied by a weighting function. → *Fourier transform*
- In a great leap of imagination, Fourier outlined these results in a memoir in 1807 and published them in *La Theorie Analitique de la Chaleur* (The Analytic theory of Heat) in 1822. The book was translated into English in 1878.

Background (2)

- The Fourier transform is more useful than the Fourier series in most practical problems since it handles signals of finite duration.
- The Fourier transform takes us between the spatial and frequency domains.
- It permits for a dual representation of a signal that is amenable for filtering and analysis.
- Revolutionized the field of signal processing.

Example

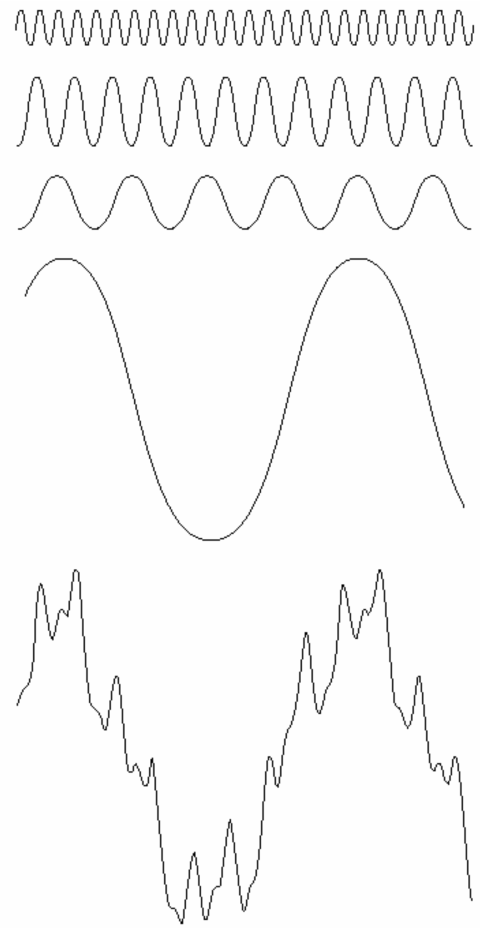
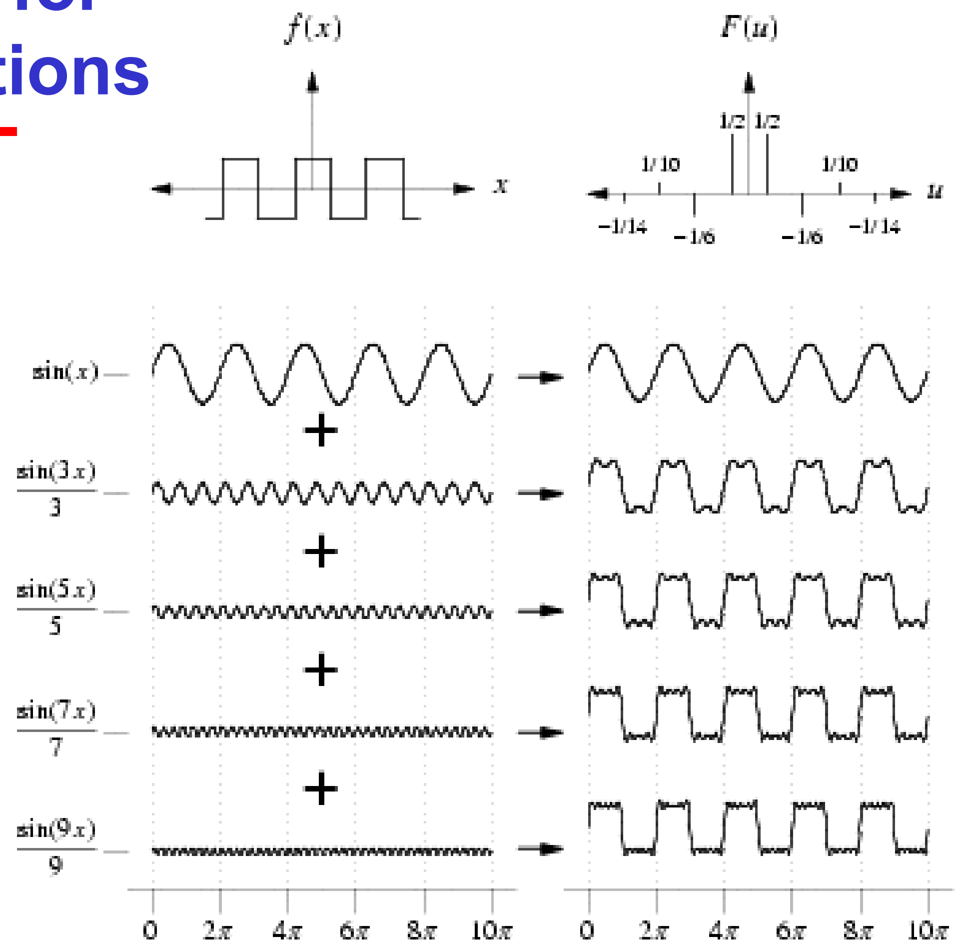


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

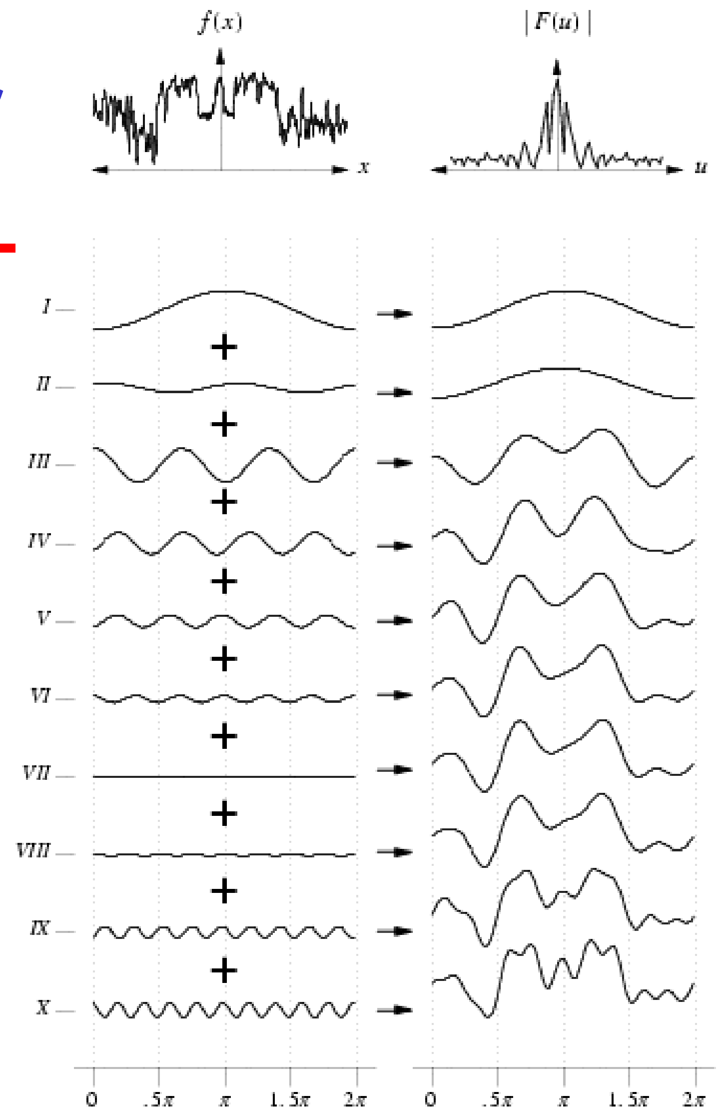
Useful Analogy

- A glass prism is a physical device that separates light into various color components, each depending on its wavelength (or frequency) content.
- The Fourier transform is a mathematical prism that separates a function into its frequency components.

Fourier Series for Periodic Functions



Fourier Transform for Aperiodic Functions



Fourier Analysis and Synthesis

- Fourier analysis: determine amplitude & phase shifts
- Fourier synthesis: add scaled and shifted sinusoids together
- Fourier transform pair:

$$\text{Forward F.T.} \quad F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

$$\text{Inverse F.T.} \quad f(x) = \int_{-\infty}^{\infty} F(u) e^{+i2\pi ux} dx$$

where $i = \sqrt{-1}$, and

$$\underset{\substack{\uparrow \\ \text{Euler's formula}}}{\mathrm{e}^{\pm i2\pi ux}} = \cos(2\pi ux) \pm i \sin(2\pi ux) \leftarrow \text{complex exponential at freq. } u$$

Fourier Coefficients

- Fourier coefficients *F(u)* specify, for each frequency *u*, the amplitude and phase of each complex exponential.
- *F(u)* is the frequency spectrum.
- *f(x)* and *F(u)* are two equivalent representations of the same signal.

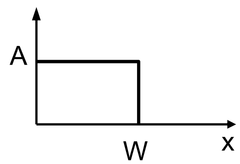
$$F(u) = R(u) + iI(u)$$

$$|F(u)| = \sqrt{R^2(u) + I^2(u)} \leftarrow \text{ magnitude spectrum; aka Fourier spectrum}$$

$$\Phi(u) = \tan^{-1}\frac{I(u)}{R(u)} \leftarrow \text{phase spectrum}$$

$$\begin{aligned} P(u) &= |F(u)|^2 \\ &= R^2(u) + I^2(u) \leftarrow \text{ spectral density} \end{aligned}$$

1D Example



$$f(x)=\begin{cases} A & 0\le x\le W \\ 0 & x>W \end{cases}$$

$$F(u)=\int_{-\infty}^{\infty} f(x)e^{-i2\pi ux}dx \qquad \text{note}: \int e^{ax}dx=\frac{1}{a}e^{ax}$$

$$F(u)=\int_{0}^{W} Ae^{-i2\pi ux}dx$$

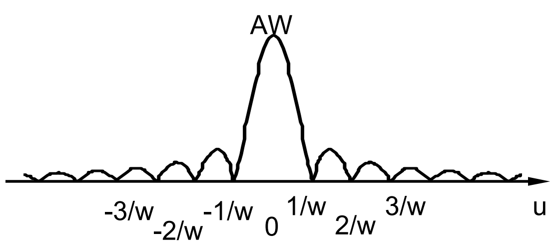
$$F(u)=\frac{-A}{i2\pi u}\left[e^{-i2\pi ux}\right]_0^W=\frac{-A}{i2\pi u}\left[e^{-i2\pi uw}-1\right]=\frac{A}{i2\pi u}[1-e^{-i2\pi uw}]$$

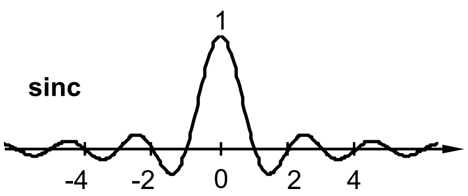
$$F(u)=\frac{A}{i2\pi u}[e^{i\pi uw}-e^{-i\pi uw}]e^{-i\pi uw} \qquad \text{note}: \sin x=\frac{e^{ix}-e^{-ix}}{2i}$$

$$F(u)=\frac{A}{\pi u}\sin(\pi uW)e^{-i\pi uW} \leftarrow \textit{complex function}$$

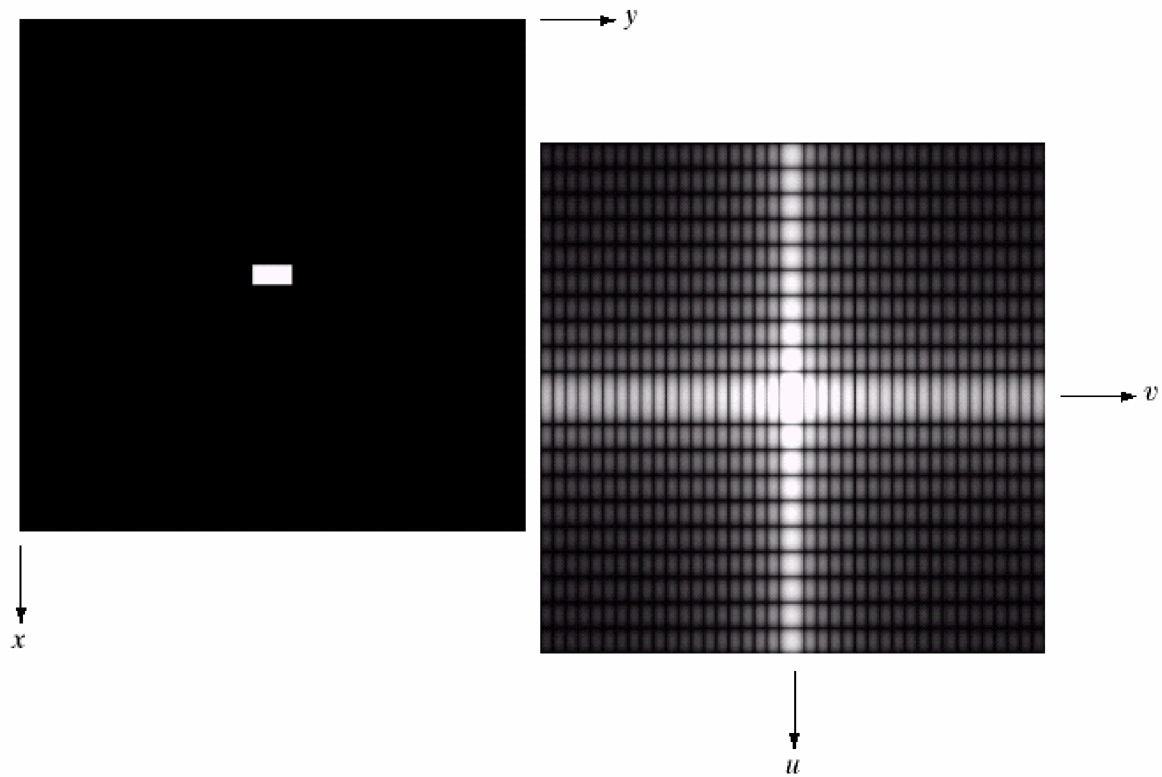
$$|F(u)|=\left|\frac{A}{\pi u}\right|\left|\sin(\pi uW)\right|\left|e^{-i\pi uW}\right|=AW\left|\frac{\sin(\pi uW)}{\pi uW}\right|=AW\left|\operatorname{sinc}(\pi uW)\right|$$

where $\operatorname{sinc}(x)=\dfrac{\sin(\pi x)}{\pi x}$ (*some books have* $\operatorname{sinc}(x)=\dfrac{\sin(x)}{x}$)





2D Example



a b

FIGURE 4.3 (a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

Fourier Series (1)

For periodic signals, we have the Fourier series:

$$f(x) = \sum_{n=-\infty}^{n=\infty} c(nu_0) e^{i2\pi u_0 x} \text{ where } c(nu_0) \text{ is the } n^{th} \text{ Fourier coefficient}$$

$$c(nu_0) = \frac{1}{x_0} \int_{-x_0/2}^{x_0/2} f(x) e^{-i2\pi u_0 x} dx$$

That is, the periodic signal contains all the frequencies that are harmonics of the fundamental frequency.

Fourier Series (2)

$$c(nu_0) = \frac{1}{x_0} \int_{-x_0/2}^{x_0/2} f(x) e^{-i2\pi nu_0 x} dx = \frac{1}{x_0} \int_{-W/2}^{W/2} A e^{-i2\pi nu_0 x} dx$$

$$c(nu_0) = \frac{A}{-i2\pi nu_0 x_0} (e^{-i\pi nu_0 W} - e^{+i\pi nu_0 W})$$

$$c(nu_0) = \frac{A}{\pi n} \sin(\pi nu_0 W) \leftarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}; \; u_0 x_0 = 1$$

$$c(nu_0) = \frac{Au_0 W}{\pi nu_0 W} \sin(\pi nu_0 W) = Au_0 W \mathrm{sinc}(\pi nu_0 W)$$

Note that if $\frac{W}{2} = \frac{x_0}{2}$, then we have a square wave and

$$c(nu_0) = Au_0 x_0 \mathrm{sinc}(\pi nu_0 x_0)$$

$$c(nu_0) = \begin{cases} A\,\mathrm{sinc(n)} & n = \pm 1, \pm 3, \cdots \\ 0 & n = 0, \pm 2, \pm 4, \cdots \end{cases}$$

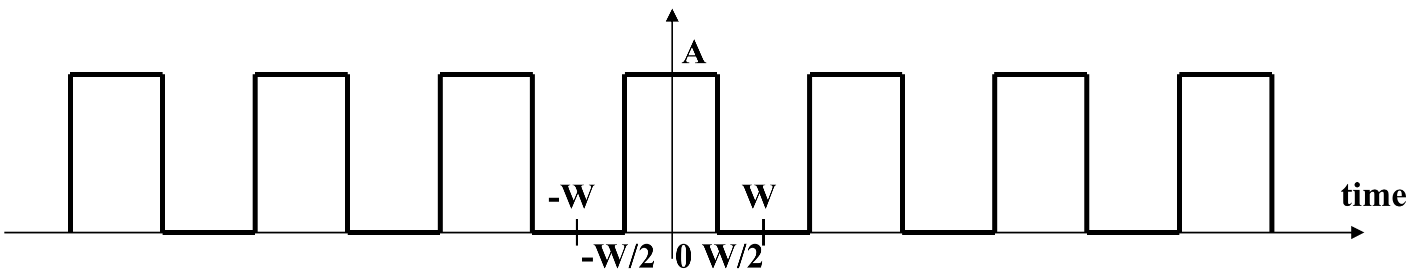
Fourier Series (3)

- The Fourier transform is applied for aperiodic signals.
- It is represented as an integral over a continuum of frequencies.
- The Fourier Series is applied for periodic signals.
- It is represented as a summation of frequency components that are integer multiples of some fundamental frequency.

Example

Ex : Rectangular Pulse Train

$$f(x) = \begin{cases} A & |x| < \dfrac{W}{2} \\ 0 & |x| > \dfrac{W}{2} \end{cases} \quad \text{in interval} [\text{-W/2, W/2}]$$



Discrete Fourier Transform

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-i2\pi \frac{ux}{N}} \qquad \text{forward DFT}$$

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{+i2\pi \frac{ux}{N}} \qquad \text{inverse DFT}$$

for $0 \le u \le N-1$ and $0 \le x \le N-1$ where N is the number of equi-spaced input samples.

The 1/N factor can be in front of $f(x)$ instead.

Fourier Analysis Code

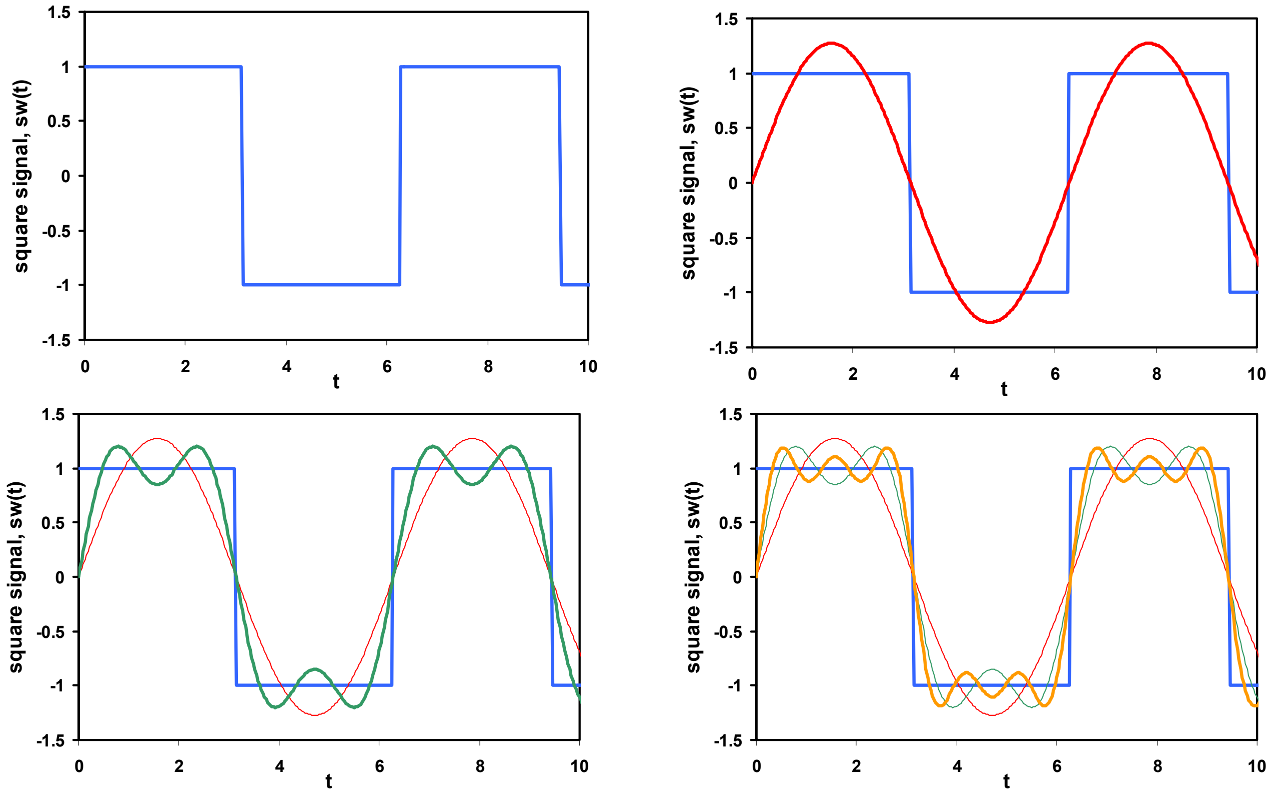
- DFT maps N input samples of f into the N frequency terms in F.

```
for(u=0; u<N; u++) { /*compute spectrum over all freq. u */
 real = imag = 0;    /*reset real, imag component of F(u)*/
 for(x=0; x<N; x++) { /* visit each input pixel */
      real += (f[x]*cos(-2*PI*u*x/N));
      imag += (f[x]*sin(-2*PI*u*x/N));
      /* Note: if f is complex, then
      real += (fr[x]*cos()-fi[x]*sin());
      imag += (fr[x]*sin()+fi[x]*cos());
      because (f_r+if_i)(g_r+ig_i)=(f_rg_r-f_ig_i)+i(f_ig_r+f_rg_i)
      */
 }
 Fr[u] = real / N;
 Fi[u] = imag / N;
}
```

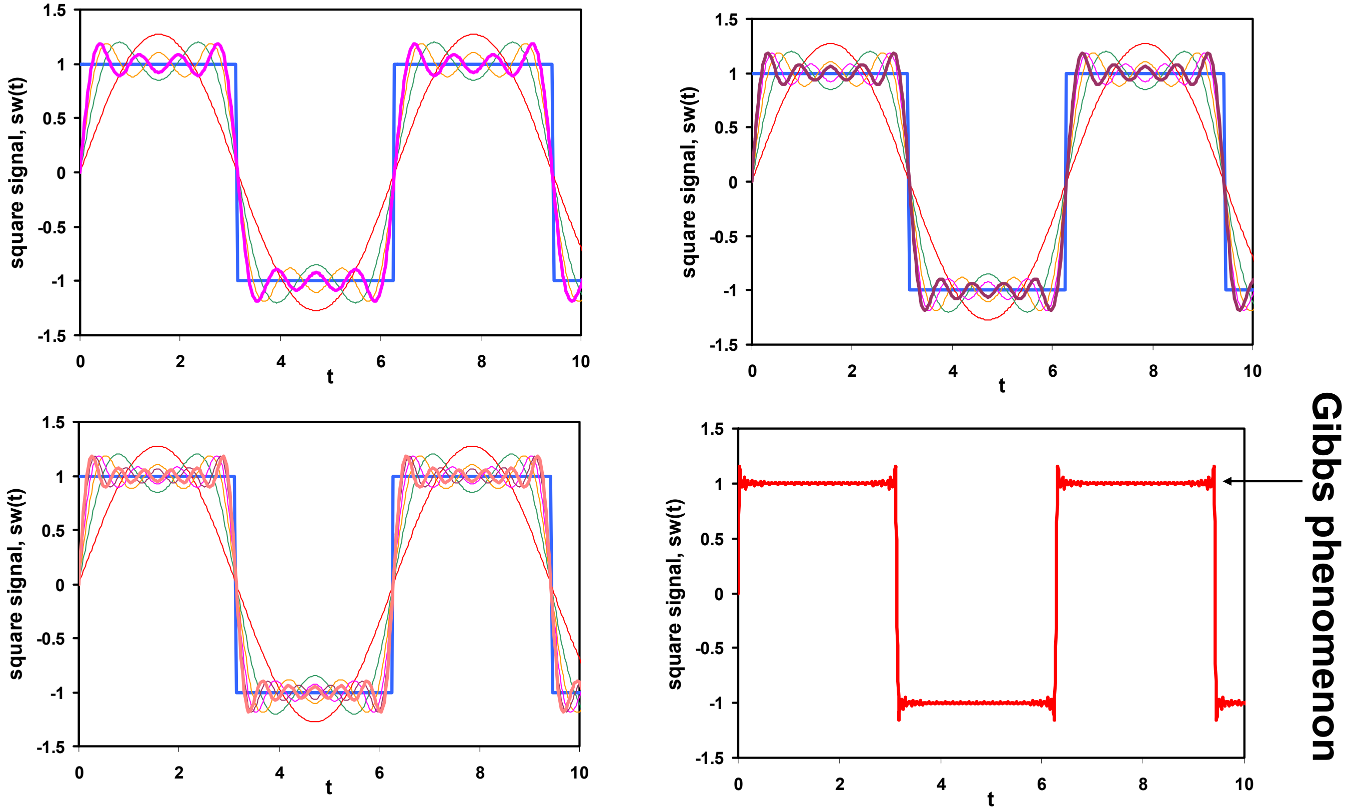
Fourier Synthesis Code

```
for(x=0; x<N; x++) {/* compute each output pixel      */
 real = imag = 0;   /* reset real, imaginary component */
 for(u=0; u<N; u++) {
      c = cos(2*PI*u*x/N);
      s = sin(2*PI*u*x/N);
      real += (Fr[u]*c-Fi[u]*s);
      imag += (Fr[u]*s+Fi[u]*c);
 }
 fr[x] = real; /* OR f[x] = sqrt(real*real + imag*imag);
 fi[x] = imag;
}
```

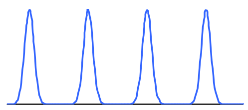
Example: Fourier Analysis (1)

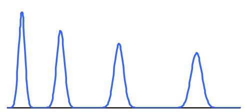


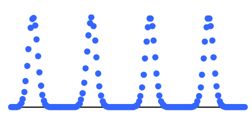
Example: Fourier Analysis (2)

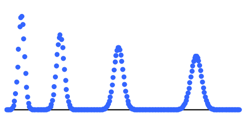


Summary









Continuous	Periodic ***(period T)***	**FS**	Discrete	$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-ik\omega t} dt$
Continuous	Aperiodic	**FT**	Continuous	$S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-i2\pi f t} dt$
Discrete	Periodic ***(period T)***	**DFS**	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-i\frac{2\pi k n}{N}}$
Discrete	Aperiodic	**DTFT**	Continuous	$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-i2\pi f n}$
Discrete	Aperiodic	**DFT**	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-i\frac{2\pi k n}{N}}$

Note: $i=\sqrt{-1}$, $\omega = 2\pi/T$, $s[n]=s(t_n)$, ***N = # of samples***

2D Fourier Transform

Continuous:

$$F\{f(x,y)\} = F(u,v) = \iint f(x,y)e^{-i2\pi(ux+vy)}dx = \iint f(x,y)e^{-i2\pi ux}e^{-i2\pi vy}dx$$

$$F^{-1}\{F(u,v)\} = f(x,y) = \iint F(u,v)e^{+i2\pi(ux+vy)}dx = \iint F(u,v)e^{+i2\pi ux}e^{+i2\pi vy}dx$$

Separable: F(u, v) = F(u)F(v)

Discrete:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x,y)e^{-i2\pi(\frac{ux}{N}+\frac{vy}{M})}dx = \frac{1}{M}\sum_{y=0}^{M-1}\left[\frac{1}{N}\sum_{x=0}^{N-1} f(x,y)e^{-i2\pi(\frac{ux}{N})}\right]e^{-i2\pi(\frac{vy}{M})}$$

$$f(x,y) = \sum_{x=0}^{N-1}\sum_{y=0}^{M-1} F(u,v)e^{+i2\pi(\frac{ux}{N}+\frac{vy}{M})}dx = \sum_{y=0}^{M-1}\left[\sum_{x=0}^{N-1} F(u,v)e^{+i2\pi(\frac{ux}{N})}\right]e^{+i2\pi(\frac{vy}{M})}$$

Separable Implementation

$$F(u,v) = \frac{1}{N}\sum_{y=0}^{N-1} e^{-j2\pi vy/N} \boxed{\frac{1}{M}\sum_{x=0}^{M-1} f(x,y)e^{-j2\pi ux/M}}$$

transform each row

$$= \frac{1}{N}\sum_{y=0}^{N-1} F(u,y)e^{-j2\pi vy/N}$$

transform each column of intermediate result

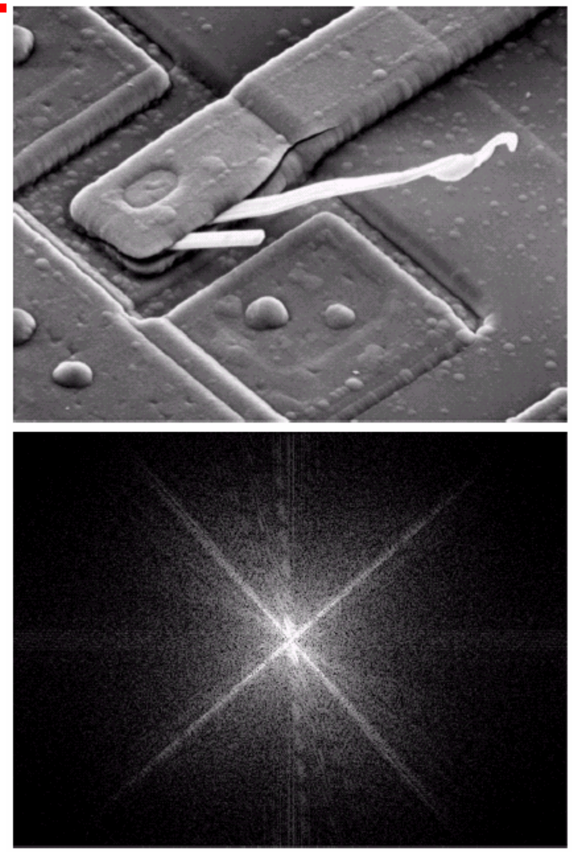
where $F(u,y) = \frac{1}{M}\sum_{x=0}^{M-1} f(x,y)e^{-j2\pi ux/M}$

The 2D Fourier transform is computed in two passes:

1) Compute the transform along each row independently.
2) Compute the transform along each column of this intermediate result.

Properties

- Edge orientations in image appear in spectrum, rotated by 90°.
- 3 orientations are prominent: 45°, -45°, and nearly horizontal long white element.

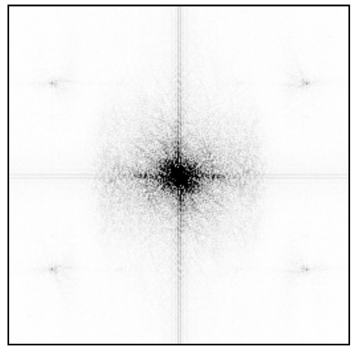


a
b

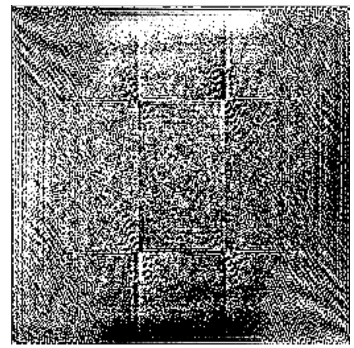
FIGURE 4.4 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Magnitude and Phase Spectrum

Mad.bw

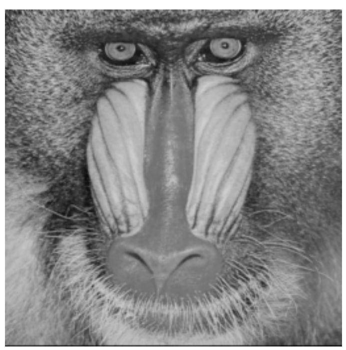


Magnitude

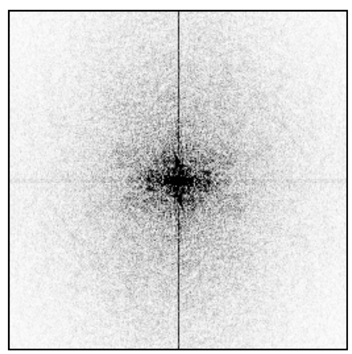


Phase

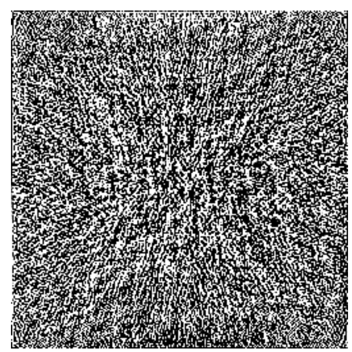
2D-Fourier transforms example



Mandrill.bw



Magnitude

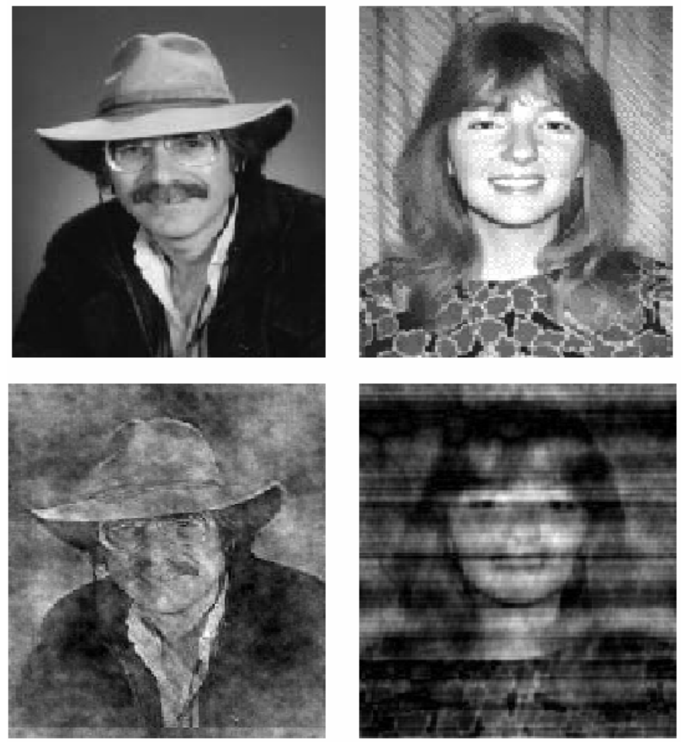


Phase

Role of Magnitude vs Phase (1)

Rick *Linda*

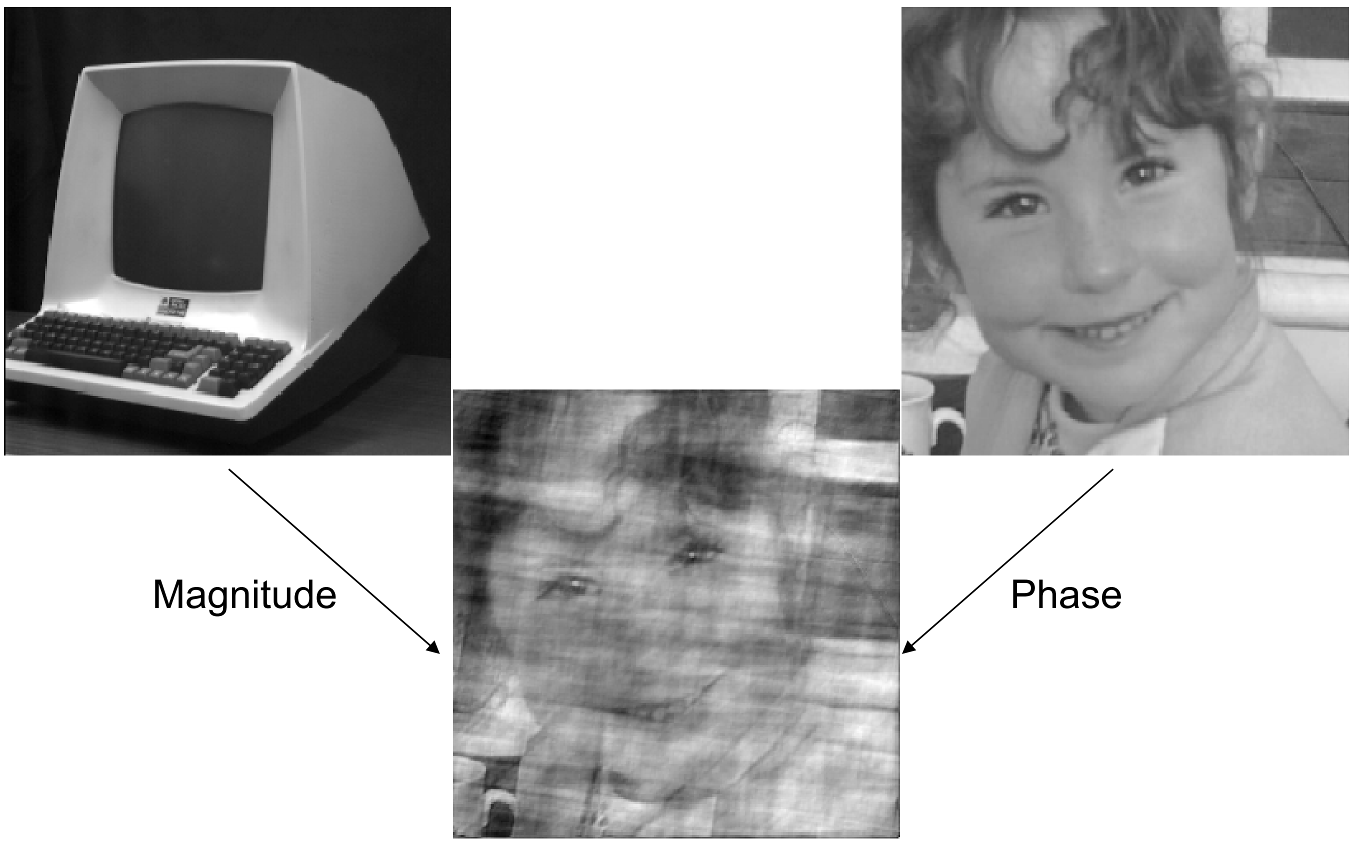
Pictures reconstructed using the Fourier phase of another picture



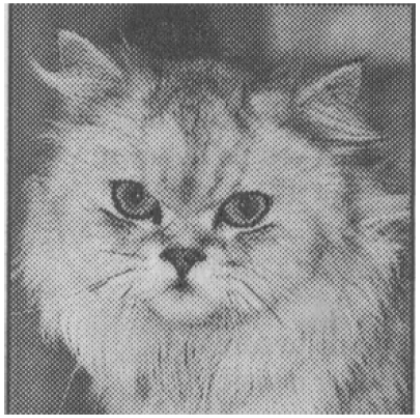
Mag{Linda}
Phase{Rick}

Mag{Rick}
Phase{Linda}

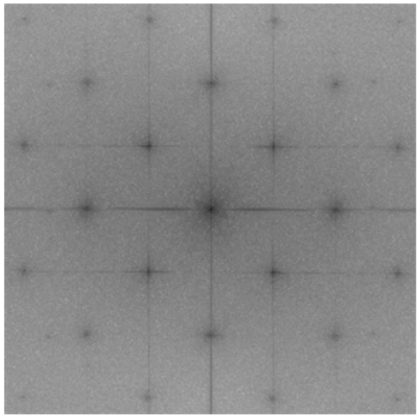
Role of Magnitude vs Phase (2)



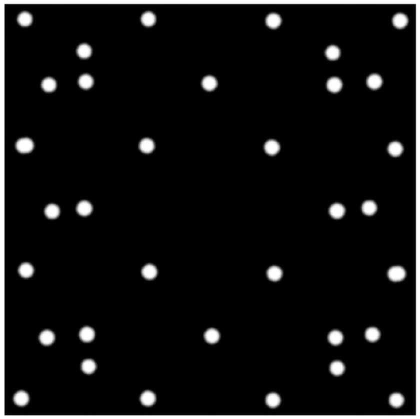
Noise Removal



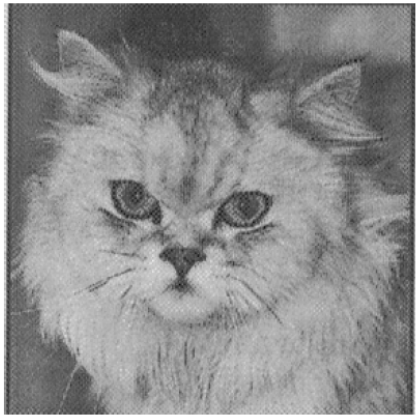
Original with noise patterns



Power spectrum showing noise spikes



Mask to remove periodic noise



Inverse FT with periodic noise removed

Fast Fourier Transform (1)

- The DFT was defined as:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)e^{-i2\pi\frac{ux}{N}} \qquad 0 \le x \le N-1$$

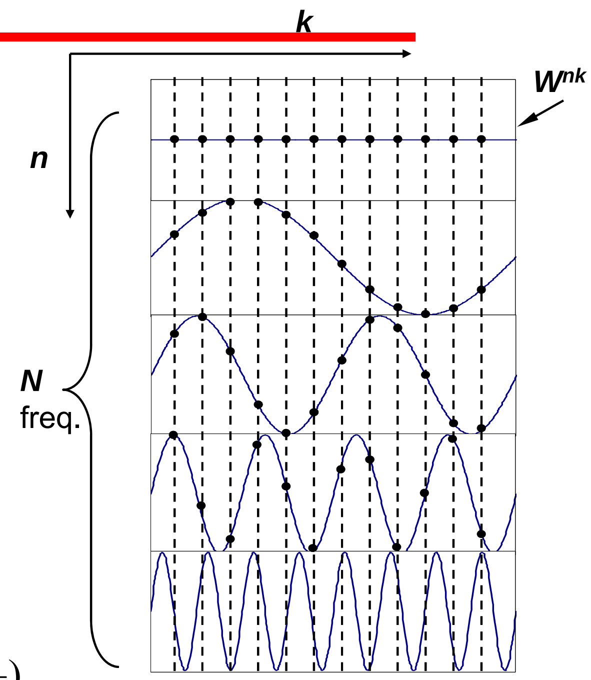
Rewrite:

$$F_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-i2\pi\frac{nk}{N}} \qquad 0 \le n \le N-1$$

$$\text{Let } F_n = \sum_{k=0}^{N-1} f_k W^{nk}$$

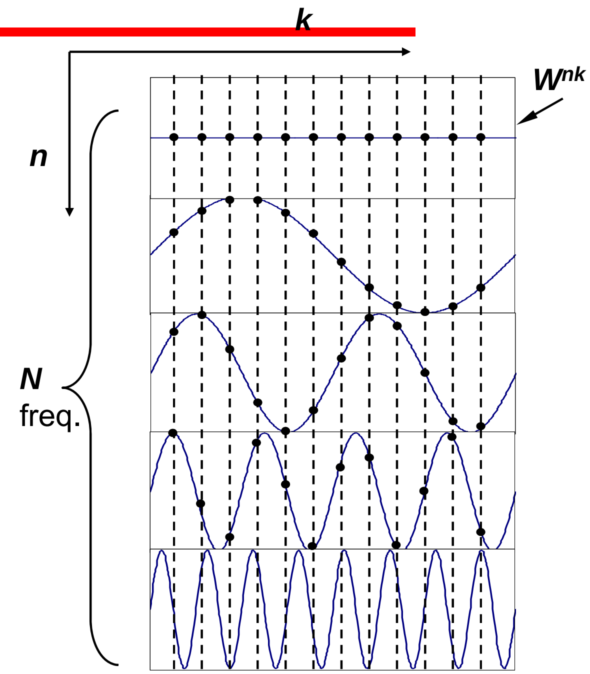
$$\text{where } W = e^{\frac{-i2\pi}{N}} = \cos\left(\frac{-2\pi}{N}\right) + i\sin\left(\frac{-2\pi}{N}\right)$$

$$\text{Also, Let } N = 2^r \text{ (} N \text{ is apower of 2)}$$



Fast Fourier Transform (2)

- W^{nk} can be thought of as a 2D array, indexed by n and k.
- It represents N equispaced values along a sinusoid at each of N frequencies.
- For each frequency n, there are N multiplications (N samples in sine wave of freq. n). Since there are N frequencies, DFT: $O(N^2)$
- With the FFT, we will derive an $O(N \log N)$ process.



Computational Advantage

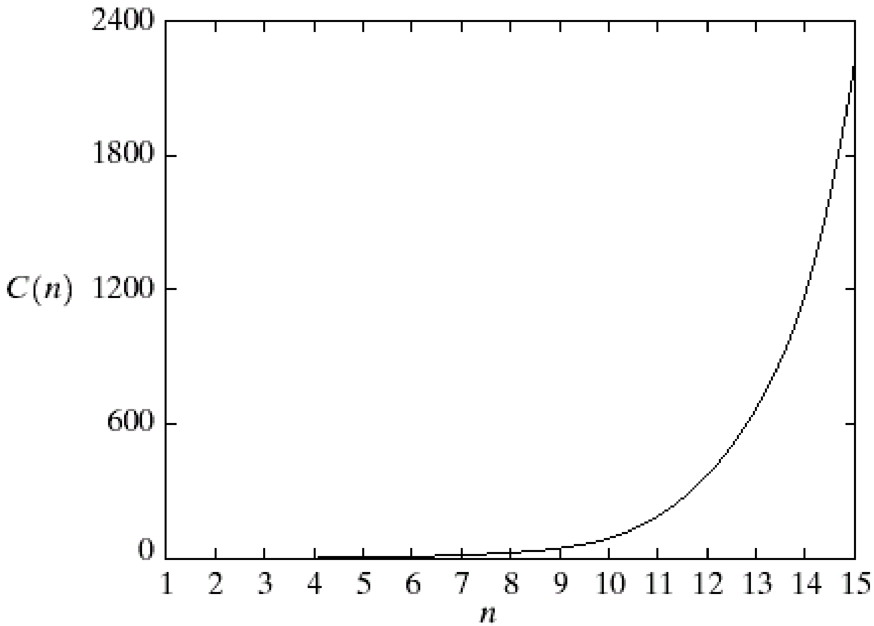


FIGURE 4.42 Computational advantage of the FFT over a direct implementation of the 1-D DFT. Note that the advantage increases rapidly as a function of n.

Danielson–Lanczos Lemma (1)

1942:

$$F_n = \sum_{k=0}^{N-1} f_k e^{-i2\pi\frac{nk}{N}}$$

$$F_n = \sum_{k=0}^{\frac{N}{2}-1} f_{2k} e^{-i2\pi\frac{n(2k)}{N}} + \sum_{k=0}^{\frac{N}{2}-1} f_{2k+1} e^{-i2\pi\frac{n(2k+1)}{N}}$$

Even Numbered Terms — Odd Numbered Terms

$f_0, f_2, f_4, \cdots$ — $f_1, f_3, f_5, \cdots$

Danielson–Lanczos Lemma (2)

$$F_n = \sum_{k=0}^{\frac{N}{2}-1} f_{2k} e^{\frac{-i2\pi nk}{N/2}} + W^n \sum_{k=0}^{\frac{N}{2}-1} f_{2k+1} e^{\frac{-i2\pi nk}{N/2}}$$

$$W = e^{\frac{-i2\pi}{N}}$$

$$F_n = F_n^e + W^n F_n^o$$

n^{th} component of F.T. of length *N/2* formed from the **even** components of *f*

n^{th} component of F.T. of length *N/2* formed from the **odd** components of *f*

Divide-and-Conquer solution: Solving a problem (F_n) is reduced to 2 smaller ones.

Potential Problem: n in F_n^e and F_n^o is still made to vary from 0 to N-1. Since each sub-problem is no smaller than original, it appears wasteful.

Solution: Exploit symmetries to reduce computational complexity.

Danielson–Lanczos Lemma (3)

Given : a DFT of length $N, F_{n+N} = F_n$

$$\text{Proof}: F_{n+N} = \sum_{k=0}^{N-1} f_k e^{\frac{-i2\pi(n+N)k}{N}} = \sum_{k=0}^{N-1} f_k e^{\frac{-i2\pi nk}{N}} e^{\frac{-i2\pi Nk}{N}} =$$

$$= \sum_{k=0}^{N-1} f_k e^{\frac{-i2\pi nk}{N}} (\cos 2\pi k - i \sin 2\pi k) = \sum_{k=0}^{N-1} f_k e^{\frac{-i2\pi nk}{N}} = F_n$$

$$W^{n+\frac{N}{2}} = \cos\left(\frac{-2\pi}{N}\left(n+\frac{N}{2}\right)\right) + i\sin\left(\frac{-2\pi}{N}\left(n+\frac{N}{2}\right)\right)$$

$$= \cos\left(\frac{-2\pi n}{N} - \pi\right) + i\sin\left(\frac{-2\pi n}{N} - \pi\right)$$

$$= -\cos\left(\frac{-2\pi n}{N}\right) - i\sin\left(\frac{-2\pi n}{N}\right)$$

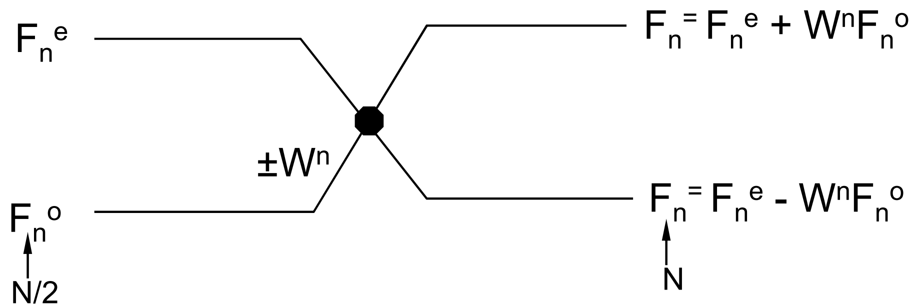
$$= -W^n$$

Main Points of FFT

$$F_n = \sum_{k=0}^{N-1} f_k e^{-i2\pi \frac{nk}{N}}$$

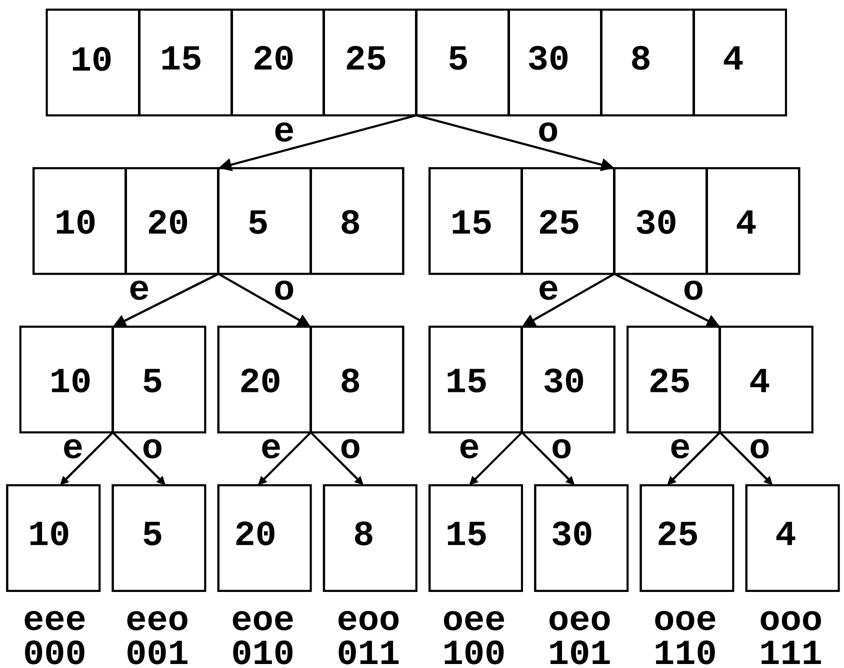
$$\begin{array}{ccccc} F_n & = & F_n^e & + & W^n F_n^o \\ \text{length } N & & \text{length } \frac{N}{2} & & \text{length } \frac{N}{2} \end{array}$$

$$\text{but } 0 \le n < N$$



FFT Example (1)

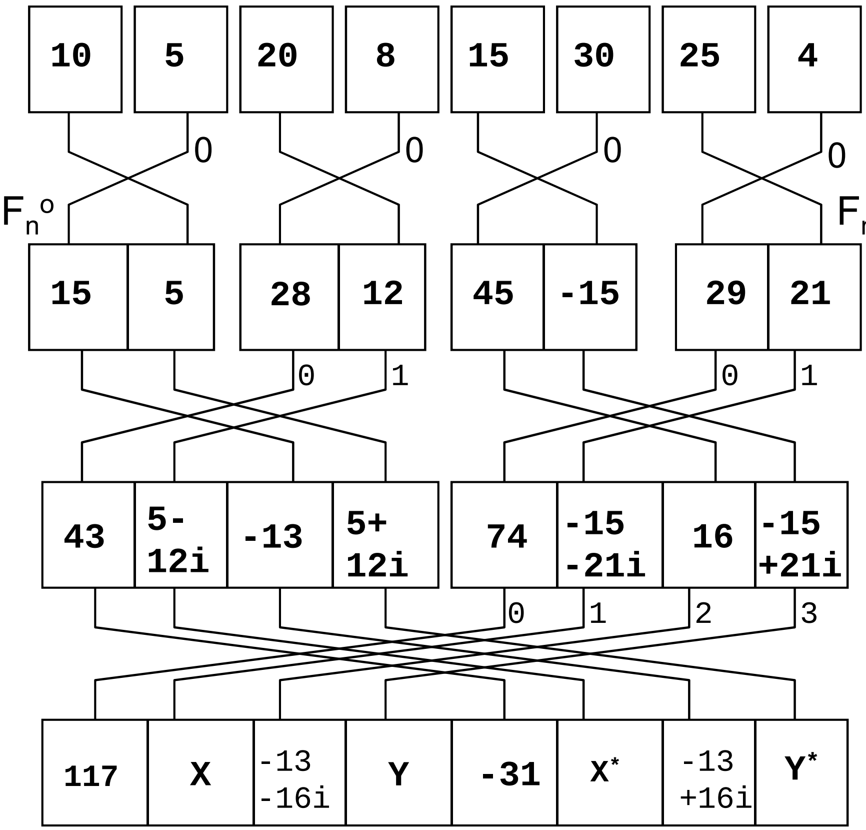
- Input: 10, 15, 20, 25, 5, 30, 8, 4



FFT Example (2)

$F_n = F_n^e + W^n F_n^o$

$F_n = F_n^e - W^n F_n^o$



$W^1 = e^{-i2\pi/4} = -i$

$W^1 = e^{-i2\pi/8} = B$

$W^2 = e^{-i4\pi/8} = C$

$W^3 = e^{-i6\pi/8} = D$

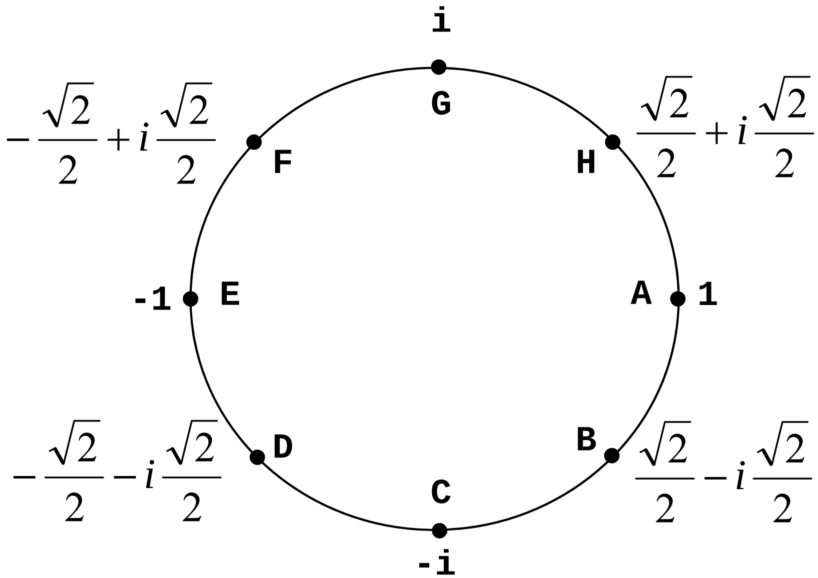
(see next page for weights B, C, D)

X=(5-12i)+(-15-21i)B

Y=(5+12i)+(-15+21i)D

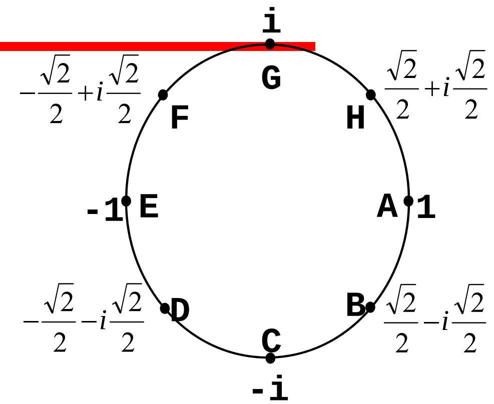
Weights

- DFT is a convolution with kernel values $e^{-i2\pi ux/N}$
- These values are derived from a unit circle.



DFT Example (1)

- Input: 10, 15, 20, 25, 5, 30, 8, 4



$$F_n = \sum_{k=0}^{N-1} f_k e^{-i2\pi\frac{nk}{N}}$$

$$F_0 = 10 + 15 + 20 + 25 + 5 + 30 + 8 + 4 = 117$$

$$F_1 = 10e^{-i2\pi(1)(0)/8} + 15e^{-i2\pi(1)(1)/8} + 20e^{-i2\pi(1)(2)/8} + \ldots + 8e^{-i2\pi(1)(6)/8} + 4e^{-i2\pi(1)(7)/8}$$

$$= 10A + 15B + 20C + 25D + 5E + 30F + 8G + 4H$$

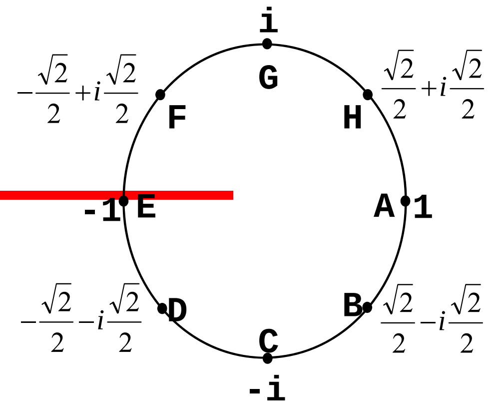
$$F_2 = 10e^{-i2\pi(2)(0)/8} + 15e^{-i2\pi(2)(1)/8} + 20e^{-i2\pi(2)(2)/8} + \ldots + 8e^{-i2\pi(2)(6)/8} + 4e^{-i2\pi(2)(7)/8}$$

$$= 10A + 15C + 20E + 25G + 5A + 30C + 8E + 4G = -13 - 16i$$

$$F_3 = 10e^{-i2\pi(3)(0)/8} + 15e^{-i2\pi(3)(1)/8} + 20e^{-i2\pi(3)(2)/8} + \ldots + 8e^{-i2\pi(3)(6)/8} + 4e^{-i2\pi(3)(7)/8}$$

$$= 10A + 15D + 20G + 25B + 5E + 30H + 8C + 4F$$

DFT Example (2)



$$F_4 = 10e^{-i2\pi(4)(0)/8} + 15e^{-i2\pi(4)(1)/8} + 20e^{-i2\pi(4)(2)/8} + \ldots + 8e^{-i2\pi(4)(6)/8} + 4e^{-i2\pi(4)(7)/8}$$
$$= 10A + 15E + 20A + 25E + 5A + 30E + 8A + 4E = -31$$
$$F_5 = 10e^{-i2\pi(5)(0)/8} + 15e^{-i2\pi(5)(1)/8} + 20e^{-i2\pi(5)(2)/8} + \ldots + 8e^{-i2\pi(5)(6)/8} + 4e^{-i2\pi(5)(7)/8}$$
$$= 10A + 15F + 20C + 25H + 5E + 30B + 8G + 4D$$
$$F_6 = 10e^{-i2\pi(6)(0)/8} + 15e^{-i2\pi(6)(1)/8} + 20e^{-i2\pi(6)(2)/8} + \ldots + 8e^{-i2\pi(6)(6)/8} + 4e^{-i2\pi(6)(7)/8}$$
$$= 10A + 15G + 20E + 25C + 5A + 30G + 8E + 4C = -13 + 16i$$
$$F_7 = 10e^{-i2\pi(7)(0)/8} + 15e^{-i2\pi(7)(1)/8} + 20e^{-i2\pi(7)(2)/8} + \ldots + 8e^{-i2\pi(7)(6)/8} + 4e^{-i2\pi(7)(7)/8}$$
$$= 10A + 15H + 20G + 25F + 5E + 30D + 8C + 4B$$

TABLE 4.1
Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$
Inverse Fourier transform	$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$
Polar representation	$F(u, v) = \lvert F(u, v) \rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u, v) \rvert = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$
Power spectrum	$P(u, v) = \lvert F(u, v) \rvert^2$
Average value	$\bar{f}(x, y) = F(0, 0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$
Translation	$f(x, y) e^{j2\pi(u_0 x/M + v_0 y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v) e^{-j2\pi(ux_0/M + vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$

Conjugate symmetry	$F(u,v) = F^*(-u,-v)$ $\lvert F(u,v)\rvert = \lvert F(-u,-v)\rvert$
Differentiation	$\dfrac{\partial^n f(x,y)}{\partial x^n} \Leftrightarrow (ju)^n F(u,v)$ $(-jx)^n f(x,y) \Leftrightarrow \dfrac{\partial^n F(u,v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x,y) \Leftrightarrow -(u^2+v^2)F(u,v)$
Distributivity	$\Im[f_1(x,y)+f_2(x,y)] = \Im[f_1(x,y)] + \Im[f_2(x,y)]$ $\Im[f_1(x,y)\cdot f_2(x,y)] \neq \Im[f_1(x,y)]\cdot\Im[f_2(x,y)]$
Scaling	$af(x,y) \Leftrightarrow aF(u,v),\ f(ax,by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert}F(u/a, v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$
Periodicity	$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$ $f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

TABLE 4.1
(continued)

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
Correlation†	$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
Convolution theorem†	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem†	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

TABLE 4.1
(continued)

Some useful FT pairs:

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\mathrm{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{1}{2}\left[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)\right]$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow j\frac{1}{2}\left[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)\right]$

† Assumes that functions have been extended by zero padding.

TABLE 4.1
(continued)