Problem 1: In class we saw how to compute all-pairs shortest paths in a graph $G$ by computing $W^{n-1}$ where $W$ is the (weighted) adjacency matrix and $W^k$ is defined as $W \otimes \ldots \otimes W$ ($k$ times) where $\otimes$ is the row-by-column matrix "multiplication" operation we saw in class (and described by pseudocode on page 624 of the textbook). Prove that this multiplication operation is associative (which is necessary to utilize the repeated squaring technique to compute $W^{n-1}$ with $O(\log n)$ multiplications).

Solution: "Traditional" matrix multiplication is associative because the integer operations of addition and multiplication are associative themselves, and moreover multiplication has the distributive property with respect to addition, i.e.

$$a(b + c) = ab + ac$$

So to prove that the matrix "multiplication" operation used by the all-pairs shortest path algorithm is associative it is sufficient to prove the above for the operations that are used inside. Recall that in this operation we have that the MIN operator plays the role of addition, and addition plays the role of multiplication.

We already know that addition is associative. We then note that the MIN operator is associative, i.e.

$$\text{MIN}(a, \text{MIN}(b, c)) = \text{MIN}(\text{MIN}(a, b), c)$$

Finally we note that addition has the distributive property with respect to MIN, since

$$a + \text{MIN}(b, c) = \text{MIN}(a + b, a + c)$$

Problem 2: What happens when you run Johnson’s algorithm on a graph that has all positive weights on its edges? What are the values of the functions $h(v)$ and $\hat{w}(e)$ for any node $v$ and edge $e$?

Solution: Nothing happens! Recall that Johnson’s algorithm adds a node $s$ to the graph and edges $(s, v)$ for all nodes $v \in V$. Those edges will be given weight 0. Then the function $h(v)$ is defined as the shortest distance from $s$ to $v$ in the new graph. But since all other edges are positive, we have that the shortest path is the new edge that we added with weight 0, so $h(v) = 0$. Remember then that if $e = (u, v)$ we have that

$$\hat{w}(e) = w(e) + h(u) - h(v)$$

and therefore $\hat{w}(e) = w(e)$ in this case.

Problem 3: In class we saw how to compute the transitive closure of a graph using a variation of the Floyd-Warshall algorithm that uses Boolean values and operations instead of the integer operation over the weights. Show how to compute the transitive closure using the same type of variation in the "matrix multiplication" algorithm to compute all-pairs shortest paths.

Solution: It is sufficient to change inf to 0, the MIN operator to OR and addition to AND.