BOTT TOWERS, COMPLETE INTEGRABILITY, AND THE EXTENDED CHARACTER OF REPRESENTATIONS

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ABSTRACT. We study certain manifolds with completely integrable torus actions, which we call Bott towers. We use these to construct *extended characters* for representations of compact Lie groups, in which all multiplicities are ± 1. As a corollary, the dimension of [the multiplicity of a weight in] a representation is obtained as the (signed) number of lattice points inside a "twisted cube" [intersected with an affine plane]. We obtain an explicit formula for the extended character which implies Demazure's character formula and which is formally similar to formulas of Littelman and Kashiwara on the crystal basis. The above results have symplectic counterparts which shed new light on the polynomial nature of the Duistermaat-Heckman measure for coadjoint orbits.

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Contents

Introduction

Two themes are interwoven throughout this paper; representation theory and symplectic geometry. These two are connected via the "orbit method" pioneered by Kirillov, Kostant, Souriau and others. Via the Borel-Weil theorem, the representations of a compact Lie group arise as spaces of holomorphic sections of line bundles over (generalized) flag varieties. In the symplectic context, flag varieties appear as coadjoint orbits. The convexity of certain projections of coadjoint orbits [Ko1] has lead to the convexity theorem in symplectic geometry [A, G-S 1]. Coadjoint orbits have also served as a model for the Duistermaat Heckman theorem [He, D-H] and for the study of symplectic fibrations in [Le, G-L-S].

Our main players are flag varieties, Bott-Samelson manifolds and certain toric varieties which we call Bott towers. We exploit the connections between these spaces in order to apply to representation theory certain results on Bott-towers.

Take a complex line bundle $\mathbf{L} \longrightarrow M$ with an action of a torus T. From this we get a *virtual character*, which is a function $\boldsymbol{\chi} : T \longrightarrow \mathbb{C}$ (see §2.7), and a *signed measure* $\boldsymbol{\sigma}$ on the vector space $\mathfrak{t}^*$ (see §1.5). These two objects are closely related, see §2.9.

Let $M = K/T$ be a flag variety with a homogeneous complex structure associated to a choice of a positive root system. If $\mathbf{L}$ is a positive line bundle then $\boldsymbol{\chi}$ coincides with the character of an irreducible representation of K (see §1.2).

If M is a *Bott-tower* (see §2.1) then both $\boldsymbol{\chi}$ and $\boldsymbol{\sigma}$ are described by a simple shape C in $\mathbb{R}^n$ which we call a *twisted cube*, see figures 1, 2. The signed measure $\boldsymbol{\sigma}$ is equal to $\pm$ Lebesgue measure on C. The virtual character is a Laurant polynomial in n variables in which all the coefficients are ± 1.

Bott-towers and flag varieties are related through *Bott-Samelson manifolds*. Demazure described a complex structure on a Bott-Samelson manifold M which relates it to a flag variety (see §3.1). Bott found an action on M of a torus of half the dimension of the space, which is not holomorphic. We describe a second complex structure, for which the action is holomorphic, and which makes the Bott-Samelson manifold into a Bott-tower (see §3.5). We then connect the two complex structures by a one parameter family of complex structures. This enables us to express the character $\boldsymbol{\chi}$ of an irreducible representation as the restriction of a virtual character $\tilde{\boldsymbol{\chi}}$ which comes from a Bott-tower. We call $\tilde{\boldsymbol{\chi}}$ an *extended character* for the representation. It depends on the choice of a reduced expression for the longest element of the Weyl group, and on no other choices.

The above result enables us to express the multiplicities in a representation as the number of points inside certain polygonal regions in $\mathbb{R}^{n-k}$, counted with signs. We also obtain a formula for the extended character which implies Demazure's character formulas and which looks similar to some formulas of Littelman and Kashiwara regarding the Crystal basis.

Analogously, we express the Heckman measure $\boldsymbol{\sigma}$ for a coadjoint orbit as a linear projection of a twisted cube. The density function for $\boldsymbol{\sigma}$ is given by the signed volume of certain polygonal regions in $\mathbb{R}^{n-k}$; this explains the piecewise polynomial nature of $\boldsymbol{\sigma}$.

All the results above continue to hold if we replace the flag varieties by *Schubert varieties*. For these too we can define a virtual character $\boldsymbol{\chi}$ and a signed measure $\boldsymbol{\sigma}$, which can again be described as projections of twisted cubes.

In the first section we review some well-known background from representation theory and from (pre-)symplectic geometry. The investigation of Bott-towers requires no background in representation theory, thus one can read §1.4, §1.5 and section 2 independently of the rest of the paper. In the third section we apply the results of section 2 to obtain the results in representation theory which were described above.

This paper has grown from the Ph.D. thesis of the first author [G]. He has defined the virtual character $\mathcal{X}$ for a Bott-tower and computed it to be a twisted cube. He used the deformation of complex structures to construct the extended character for a representation, see §2.8 for an outline of that argument.

Our results on Bott-towers have in the meantime been generalized in [K-T] to all smooth toric varieties. The corresponding character and measure are then given by shapes called twisted polytopes, which are generalizations of twisted cubes. A further generalization, to completely integrable Spin^c manifolds, will be given in [G-K].

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1. Preliminaries

As general references in representation theory and in (pre-) symplectic geometry, see [Hu], [J], [Au].

1.1. **Flag varieties and Schubert varieties.** Let G be a complex semisimple Lie group. For clarity we assume G is simply connected. Choose a Cartan subgroup H and let $\mathfrak{g} = \mathfrak{h} \oplus \sum_\alpha \mathfrak{g}_\alpha$ be the decomposition into root spaces. Denote by $\Delta \subset \mathfrak{h}^*$ the roots. Choose a set of positive roots Δ^+, with $\Delta = \Delta^+ \cup -\Delta^+$, and denote the simple roots by $\Sigma \subseteq \Delta^+$. Let B be the Borel subgroup whose Lie algebra is $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$. Let N be the normalizer of H in G, then the Weyl group is $W = N/H$. The Bruhat decomposition is $G = \bigcup_{w \in W} B\, \dot{w}\, B$ where $\dot{w}$ is a representative of w in N.

The quotient G/B is called a *(generalized) flag variety.* The Bruhat decomposition of G gives rise to a cell decomposition, $G/B = \bigcup_{w \in W} B\dot{w}B/B$. These cells are called *Bruhat cells* and the closures of these cells are called the *Schubert varieties* in G/B; we denote

$$X_w = closure(B\dot{w}B/B).$$

This variety may be singular; it decomposes into a union of Bruhat cells.

More generally, let $J \subset \Sigma$ be a subset of simple roots and let $P = P_J \subseteq G$ be the parabolic subgroup of G whose Lie algebra is $\mathfrak{p}_{\mathfrak{J}} = \sum_{\alpha \in \Delta \cap (\text{span } \mathfrak{J})} \mathfrak{g}_\alpha \oplus \mathfrak{b}$. In particular, if $J = \emptyset$ then $P = B$. The quotient G/P is also called a *(generalized) flag variety.*

Let $W^J = \{w \in W \mid w(\alpha) > 0 \text{ for all } \alpha \in J\}$. Then we have a cell decomposition $G/P = \bigcup_{w \in W^J} B\dot{w}P/P$; see [J, II,§13.8]. The Schubert varieties in G/P are the closures of the cells.

1.2. **Line bundles and multiplicities.** We now describe how the choice of an integral weight determines a holomorphic line bundle over a flag variety.

Let $K \subset G$ be a maximal compact subgroup, then we have a K-equivariant diffeomorphism $K/T \longrightarrow G/B$ where T is the maximal torus in K. Similarly, if $P \subset G$ is a parabolic subgroup then we have an equivariant diffeomorphism $K/L \longrightarrow G/P$ for a subgroup $L \subseteq K$.

Pick λ in the integral weight lattice in $i\mathfrak{t}^*$. Since G is simply connected, we have a homomorphism $e^\lambda : T \longrightarrow S^1$. Let $\mathbb{C}_\lambda$ be a one dimensional complex vector space on which T acts as multiplication by e^λ. The associated bundle, $\mathbf{E}_\lambda = K \times_T \mathbb{C}_\lambda$, is a complex line bundle over the real manifold K/T. It can be made into a *holomorphic* line bundle over the complex manifold G/B in the following way. The orthogonal projection of Lie algebras $\mathfrak{b} \longrightarrow \mathfrak{h}$ descends to a homomorphism $\Upsilon : B \longrightarrow H$ [1] of complex Lie groups [J, II,§1.8]. Let B act on $\mathbb{C}_\lambda$ as multiplication by $e^\lambda \circ \Upsilon$. Then we can write $\mathbf{E}_\lambda = G \times_B \mathbb{C}_\lambda$, which is a holomorphic line bundle over G/B.

For a singular weight λ let $J \subseteq \Sigma$ be the set of simple roots α such that $\langle \lambda, \alpha \rangle = 0$, where $\langle \cdot, \cdot \rangle$ is the Killing form on $i\mathfrak{t}^*$. Let $P \subset G$ be the corresponding parabolic subgroup. Then $e^\lambda : H \longrightarrow \mathbb{C}^\times$ extends to a homomorphism $P \longrightarrow \mathbb{C}^\times$. Consider the map $G/B \longrightarrow G/P$, then $\mathbf{E}_\lambda$ is the pull-back of a holomorphic line bundle over G/P. From now on we denote by $\mathbf{E}_\lambda$ the bundle over G/P.

The *positive Weyl chamber* in $i\mathfrak{t}^*$ is the set of $\lambda \in i\mathfrak{t}^*$ such that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta^+$. An integral weight λ is *dominant* if it lies in the closure of the positive Weyl chamber. Such λ is singular if and only if it lies on the boundary.

The group G acts holomorphically on the total space of $\mathbf{E}_\lambda$ by left multiplication. This induces a representation of G on the vector space of holomorphic sections, $\Gamma_{\text{hol}}(\mathbf{E}_\lambda)$. Denote this representation by R_λ. If λ is dominant then, by the Borel-Weil theorem [Kn], R_λ is the irreducible representation of G with a highest weight λ; otherwise $R_\lambda = \{0\}$. All irreducible finite dimensional representations of G arise in this manner.

Now, fix a Schubert variety $X_w \overset{i}{\hookrightarrow} G/B$ and consider $i^*\mathbf{E}_\lambda$; see [J, §14]. We clarify our meaning of a holomorphic section over a (singular) Schubert variety. Since the Bruhat cell $B\dot{w}B/B$ is dense in X_w and is locally closed in G/B, we can think of the holomorphic sections of i^*E_λ as being the restrictions to $B\dot{w}B/B$ of the sections of $\mathbf{E}_\lambda$. Again we have a left action, of B, which induces a representation of B on $\Gamma_{\text{hol}}(i^*\mathbf{E}_\lambda)$.

Fix a dominant integral weight λ in $i\mathfrak{t}^*$ and consider the representation R_λ of G with highest weight λ. Its character $\chi_T = \chi_T(\lambda)$ is the complex valued function on T defined by $a \mapsto \mathit{trace}(R_\lambda(a))$. We omit λ when there is no possibility of confusion.

[1] Υ is the Greek letter Upsilon

We can write $\boldsymbol{\chi}_T(a) = \sum_{\mu \in \ell^*} m_\mu e^\mu(a)$ where the summation is over the integral weight lattice $\ell^* \subset i\mathfrak{t}^*$ and where $mult(a) = m_\mu$ is the multiplicity of e^μ in the restricted representation $R_\lambda|_T$.

Similarly, if we fix a Bruhat variety $X_w \overset{i}{\hookrightarrow} G/B$ then the torus $T \subseteq B$ acts on the space of holomorphic sections of i^*E_λ and again we get a character, this time arising from the restriction to T of a representation of B.

1.3. **Symplectic structure.** Now let λ be any *regular* element in $i\mathfrak{t}^*$. The Killing metric gives an embedding $j : \mathfrak{t}^* \hookrightarrow \mathfrak{k}^*$ where $\mathfrak{k} = \mathrm{Lie}\,(\mathfrak{K})$. Let $\lambda' = (2\pi i)^{-1}\lambda$ and $j(\lambda')$ be its image in $\mathfrak{k}^*$. The group K acts on $\mathfrak{k}^*$ by the *coadjoint* action. The stabilizer of $j(\lambda')$ in K is exactly T; this is the meaning of being regular. Therefore, the map $a \mapsto a \cdot j(\lambda')$ identifies K/T with a coadjoint orbit in $\mathfrak{k}^*$. Composing with the inverse of the map $K/T \longrightarrow G/B$, we get a K-equivariant diffeomorphism between the flag manifold G/B and the coadjoint orbit of $j(\lambda')$ in $\mathfrak{k}^*$.

More generally, if $\lambda \in i\mathfrak{t}^*$ is singular then let $J = \{\alpha \in \Sigma \mid \langle \lambda, \alpha \rangle = 0\}$ as in §1.2. Then the coadjoint orbit through $j(\lambda')$ is isomorphic to the flag variety G/P_J as smooth manifolds with left K-actions.

Any coadjoint orbit has a natural K-invariant symplectic form, i.e., a closed nondegenerate differential 2-form, due to Kirillov, Kostant and Souriau. By the diffeomorphisms described above, the choice of λ induces an invariant symplectic form on a flag manifold. If λ is integral then this symplectic form coincides with the curvature of the natural connection on the bundle $\mathbf{E}_\lambda$ which was defined in §1.2.

Even if λ is not integral, the symplectic structure on X is compatible with the complex structure. Consequently, every complex submanifold of X is also a symplectic submanifold. In particular, the Bruhat cells are symplectic.

1.4. **Moment maps.** Let (M, ω) be a symplectic manifold and T a torus which acts on M and preserves ω. A *moment map* is defined to be a map $\Phi : M \longrightarrow \mathfrak{t}^*$ such that

$$\langle d\Phi, \xi \rangle = -\iota(\xi_M)\omega \quad \text{for all } \xi \in \mathfrak{t} \tag{1.1}$$

where ξ_M is the vector field on M which generates the action of the one parameter subgroup $\{\exp(t\xi)\ ,\ t \in \mathbb{R}\}$. A *presymplectic* form is just a fancy name for a closed 2-form. The above definition also makes sense when ω is presymplectic.

Note that (1.1) determines Φ up to a translation by an element of $\mathfrak{t}^*$, and that Φ always exists if M is simply connected. Also note that (1.1) implies that Φ is T-invariant and that the pullback of ω to an orbit is zero; see [Au].

If M is a coadjoint orbit for K and T is the maximal torus, acting by the coadjoint action, then the moment map Φ is the inclusion $M \hookrightarrow \mathfrak{k}^*$ followed by the projection $\mathfrak{k}^* \longrightarrow \mathfrak{t}^*$. The Bruhat cells in M are T-invariant and symplectic, and the restriction of Φ to a Bruhat cell is a corresponding moment map. In fact since the restriction makes $p \circ \Phi$ a well defined map, on all of the Schubert variety X_w we define it to be the moment map. Moreover we will treat (X_w, T, ω, Φ), as a symplectic manifold, though ω is well defined on a dense open set, the associated Bruhat cell, Φ is defined by restriction, and satisfies 1.1 on the Bruhat cell.

Here is one particularly important situation in which moment maps arise. Take a manifold M with an action of a torus T. Take a principal S^1-bundle $\pi : P \longrightarrow M$ and a lifting of the action to P. Let Θ be an invariant connection and ω its curvature. Then Θ is a 1-form on P and ω is a 2-form on M, they take values in Lie (S^1) and are related by $\pi^*\omega = d\Theta$. We identify Lie (S^1) with $\mathbb{R}$ such that the exponential map is $x \mapsto e^{2\pi i x}$. This allows us to view Θ and ω as real valued differential forms. Although ω might not be symplectic, it is always closed and invariant. Define $\Phi : M \longrightarrow \mathfrak{t}^*$ by

$$\langle \Phi \circ \pi, \xi \rangle = \langle \Theta, \xi_P \rangle \quad \text{for all } \xi \in \mathfrak{t} \tag{1.2}$$

where ξ_P is the vector field on P determined by ξ. Then Φ is a moment map for (M, T, ω).

1.5. **Duistermaat-Heckman measure.** Let M be an oriented, compact manifold of real dimension $2n$. Let T be a torus which acts on M, let ω be a T-invariant closed 2-form, and let $\Phi : M \longrightarrow \mathfrak{t}^*$ be a moment map.

Consider the Liouville measure on M; the measure of an open subset $A \subseteq M$ is defined to be $\int_A \omega^n/n!$. This is a signed measure on M. Its push-forward, $\Phi_* \omega^n/n!$, is called the *Duistermaat-Heckman* (D-H) measure. It is a signed measure on $\mathfrak{t}^*$ which is determined by (M, T, ω) up to a translation in $\mathfrak{t}^*$.

Theorem 1 (Guillemin and Sternberg)**.** *$\Phi_*\omega^n$ only depends on the cohomology class of ω in $H^2(M, \mathbb{R})$.*

Proof. This was proven by Guillemin and Sternberg in [G-S 3]. Since their paper is not published, we sketch their proof here. Suppose that ω_1 and ω_2 are cohomologous and let Φ_1, Φ_2 be corresponding moment maps. It is sufficient to show that the Fourier transforms of the push-forward measures coincide, i.e., that

$$\int_M e^{i\langle \Phi_1, \xi \rangle} \omega_1^n = \int_M e^{i\langle \Phi_2, \xi \rangle} \omega_2^n \quad \text{for all } \xi \in \mathfrak{t}. \tag{1.3}$$

Further, it is sufficient to prove (1.3) for ξ which generates a circle $S^1 \subseteq T$, because the set of such ξ's is dense. Define $\varphi_j = \langle \Phi_j, \xi \rangle$ for $j = 1, 2$. We have $\omega_2 = \omega_1 + d\beta$ and by averaging over T we can assume that β is an invariant 1-form. Then $\varphi_1 + \iota(\xi_M)\beta$ is a moment map for ω_2, so it differs from φ_2 by a constant, and we assume that it is *equal* to φ_2.

Now consider the graded ring $A^*_{S^1} \otimes \mathbb{C}[u]$ where $A^*_{S^1}$ denotes the S^1-invariant differential forms on M and $\mathbb{C}[u]$ is the ring of polynomials in the variable u, with degree$(u) = 2$. Define a differential $\tilde{d}$ by $\tilde{d}(\alpha \otimes p(u)) = d\alpha \otimes p(u) + \iota(\xi_M)\alpha \otimes up(u)$. One can explicitly check the following facts.

- $\tilde{d}^2 = 0$.
  Denote $H^*_{S^1}(M) = \ker(\tilde{d}) / \mathit{image}\,(\tilde{d})$.
- The ring structure on $A^*_{S^1} \otimes \mathbb{C}[u]$ descends to $H^*_{S^1}(M)$.
- Integration over M defines a map $\int_M : A^*_{S^1} \otimes \mathbb{C}[u] \longrightarrow \mathbb{C}[u]$ which descends to a map $H^*_{S^1}(M) \longrightarrow \mathbb{C}[u]$. If rank$(\alpha) \neq n$ then we define $\int_M \alpha = 0$.
- $\tilde{\omega_j} := \omega_j \otimes 1 + \varphi_j \otimes u$ are $\tilde{d}$-closed for $j = 1, 2$; and $\tilde{\omega}_2 = \tilde{\omega}_1 + \tilde{d}(\beta \otimes 1)$

These facts imply that $\int_M \tilde{\omega}_1^n = \int_M \tilde{\omega}_2^n$ for all n. Define $e^{\tilde{\omega}} = 1 + \tilde{\omega} + \tilde{\omega}^2/2! + \ldots$, then $\int_M e^{\tilde{\omega}_1} = \int_M e^{\tilde{\omega}_2}$. Restricting to the components of $e^{\tilde{\omega}_j}$ whose form-parts have degree n, we get

$$\int_M e^{u\varphi_1} \omega_1^n / n! = \int_M e^{u\varphi_2} \omega_2^n / n!$$

as an equality between power series in u. Setting $u = i$ gives (1.3). $\square$

The measure $\Phi_* \omega^n$ is absolutely continuous with respect to Lebesgue measure on the smallest affine space which contains *image* (Φ) (which is all of $\mathfrak{t}^*$ if the action is effective). The density function $\rho(\alpha)$ is *piecewise polynomial* by the Duistermaat-Heckman theorem [D-H]. In section 3 we will show that this remains true for Schubert varieties which are not smooth manifolds.

Remark 1.4. Let (M, T, ω, Φ) be as above. Take a homomorphism $A : T' \longrightarrow T$ where T' is a torus. Consider the action of T' on M given by the composition of A with the action of T. We have $dA : \mathfrak{t}' \longrightarrow \mathfrak{t}$, and dually, $L = dA^* : \mathfrak{t}^* \longrightarrow \mathfrak{t}'^*$. Then $\phi' = L \circ \Phi$ is a moment map for (M, T', ω) and the corresponding D-H measure in $\mathfrak{t}'^*$ is equal to the push-forward by L of the D-H measure $\Phi_* \omega^n$ in $\mathfrak{t}^*$.

1.6. **Relation between multiplicities and the Heckman measure.** Recall that the choice of a dominant weight $\lambda \in i\mathfrak{t}^*$ gives rise to a flag manifold $X = G/P$ and to two objects. The first is the character $\boldsymbol{\chi}$ of the irreducible representation R_λ with a highest weight λ. It is determined by its multiplicity function, *mult* : $\ell^* \longrightarrow \mathbb{Z}$, which sends $\mu \mapsto m(\mu, \lambda) =$ the multiplicity with which μ occurs in $R_\lambda|_T$. The second object is a measure on $\mathfrak{t}^*$ which is determined by its density function, $\rho : \mathfrak{t}^* \longrightarrow \mathbb{R}$.

For every $k \in \mathbb{N}$ we have

$$m(k\mu, k\lambda) = k^r \rho(i^{-1}\mu) + O(k^{r-1}) \tag{1.5}$$

where $r = \dim G - 2\dim T$; [He], [G-S 2]. Thus we can think of ρ as a continuous approximation to *mult*. Note that although ρ is naturally a function on $\mathfrak{t}^*$ and weights naturally live in $i\mathfrak{t}^*$ we identify $\mathfrak{t}^* \cong \mathfrak{it}^* \cong \mathbb{R}^{\dim \mathfrak{T}}$.

The statement (1.5) remains true even when applied to a Schubert variety X_w and not to the whole coadjoint orbit X. If X_w is smooth then this would follow from [G-S 2], theorems 6.2 and 6.5, and remark 1, where their O is our $\{\lambda\}$. In section 3 we shall obtain another proof, which also applies when X_w is singular.

2. Bott towers

The complex manifolds discussed in this section were shown to the first author by R. Bott and are in the same spirit as the manifolds used in the splitting principle [B-T, §21]. The underlying smooth manifolds arise in a construction of R. Bott and H. Samelson [B-S].

2.1. **What is a Bott tower?** We now construct a family of compact complex manifolds. They will have extra structure in that they will be iterated fibrations and the fiber maps will have certain distinguished sections. Take a holomorphic line bundle $\mathbf{L}_1$ over $M_1 = \mathbb{C}P^1$. Take its direct sum with the trivial bundle, and projectivize each fiber. This produces a manifold $M_2 = \mathbb{P}(\mathbf{1} \oplus \mathbf{L}_1)$ which is a bundle over M_1 with a fiber $\mathbb{C}P^1$; this is a Hirzebruch surface. We can repeat this process n times, so that each M_j is a $\mathbb{C}P^1$-bundle over M_{j-1}:

$$
\begin{array}{ccccccc}
 & & & & & \mathbb{P}(\mathbf{1}\oplus\mathbf{L}_n) & = & M_n \\
 & & & & & \downarrow \pi_n & & \\
 & & & & & M_{n-1} & & \\
 & & & & \iddots & & & \\
 & & \mathbb{P}(\mathbf{1}\oplus\mathbf{L}_2) & = & M_2 & & & \\
 & & \downarrow \pi_2 & & & & & \\
\mathbb{C}P^1 & = & M_1 & & & & & \\
\downarrow \pi_1 & & & & & & & \\
\{a\ point\} = M_0 & & & & & & &
\end{array}
$$

We think of $\mathbb{C}P^1$ as a sphere with a South pole $[1,0]$ and a North pole $[0,1]$. The zero section of $\mathbf{L}_j$ gives rise to a holomorphic section, the *South pole section*;

$$\sigma_j^S : M_{j-1} \longrightarrow M_j.$$

Similarly, we obtain the *North pole section* $\sigma_j^N : M_{j-1} \longrightarrow M_j$ by letting the first coordinate in $\mathbb{P}(\mathbf{1}\oplus\mathbf{L}_j)$ to vanish.

A *Bott tower* is by definition a collection $\{M_j, \pi_j, \sigma_j^N, \sigma_j^S\}_{j=1}^n$ which can be realized by the above process. It is thus a complex manifold M_n together with the additional fibration and section structure.

Example 2.1. $\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ (n times) is a Bott tower.

2.2. **Construction of Bott towers.** A one-step Bott tower can be written as a quotient; $M_1 = \mathbb{C}P^1 = (\mathbb{C}^2 \smallsetminus \mathbf{0})/\mathbb{C}^\times$, where $\mathbb{C}^\times$ acts diagonally. We construct a line bundle over M_1 by $\mathbf{L}_2 = (\mathbb{C}^2 \smallsetminus \mathbf{0}) \times_{\mathbb{C}^\times} \mathbb{C}$, where $\mathbb{C}^\times$ acts on $\mathbb{C}$ by $a : v \mapsto a^{-c}v$ for some integer c. In $\mathbf{L}_2$ we have $[z,w,v] = [za, wa, a^c v]$ for all $a \in \mathbb{C}^\times$. The 2-step tower $M_2 = \mathbb{P}(\mathbf{1}\oplus\mathbf{L_2})$ can be written as $M_2 = (\mathbb{C}^2 \smallsetminus \mathbf{0})^\mathbf{2}/\mathbf{G}$ where $G = (\mathbb{C}^\times)^2$ acts on the right by $(z_1, w_1, z_2, w_2)\cdot(a,b) = (z_1 a, w_1 a, z_2 b, a^c w_2 b)$.

We can continue to construct higher Bott towers in a similar way. At each step we have $\mathbf{L}_{j+1} = (\mathbb{C}^2 \smallsetminus \mathbf{0})^\mathbf{j} \times_\mathbf{G} \mathbb{C}$ for $G = (\mathbb{C}^\times)^j$, where the action of G on $\mathbb{C}$ is encoded by j integers. In this way we get an n-step tower M_n from any collection of $n(n-1)/2$ integers $\{c_{ij}\}_{1\le i<j\le n}$:

$$M_n = (\mathbb{C}^2 \smallsetminus \mathbf{0})^\mathbf{n}/\mathbf{G} \tag{2.2}$$

where $G = (\mathbb{C}^\times)^n$, and its ith factor acts on the right by

$$(z_1, w_1, \ldots, z_n, w_n)\cdot a_i = (z_1, w_1, \ldots, z_i a_i, w_i a_i, \ldots, z_j, a_i^{c_{ij}} w_j, \ldots). \tag{2.3}$$

It is easy to see that M_n is thus isomorphic to $\mathbb{P}(\mathbf{1}\oplus\mathbf{L}_n)$. We denote by $[z_1, \ldots, w_n]$ the corresponding point in M_n and we think of $z_1, \ldots, w_n$ as generalized homogeneous coordinates on M_n.

We can construct a line bundle over M_n from the integers $(l_1, \ldots, l_n)$ by

$$\mathbf{L} = (\mathbb{C}^2 \smallsetminus \mathbf{0})^\mathbf{n} \times_\mathbf{G} \mathbb{C}, \tag{2.4}$$

where the ith factor of $G = (\mathbb{C}^\times)^n$ acts by

$$((z_1, \ldots, w_n), v)\cdot a_i = ((z_1, \ldots, w_n)\cdot a_i, a_i^{l_i} v), \tag{2.5}$$

the right action of a_i on $(z_1, \ldots, w_n)$ being given in (2.3).

It will also be useful to construct Bott towers from certain compact manifolds, although this construction only yields the underlying smooth manifolds and not their complex structures. We now describe this construction.

Start with $M_1 = \mathbb{C}P^1$ as a quotient S^3/S^1 where $S^3 = \{(z,w) \in \mathbb{C}^2 \ : \ |z|^2 + |w|^2 = 1\}$ and where $S^1 = \{a \in \mathbb{C} \ : \ |a| = 1\}$ acts diagonally on (z,w). Continue in a similar way as before to yield $M_n = (S^3)^n/K$ where $K = (S^1)^n$ acts as in (2.3).

Given integers $l_1, \ldots, l_n$, construct a principal S^1-bundle $P \longrightarrow M_n$ by $P = (S^3)^n \times_K S^1$ where $K = (S^1)^n$ acts as in (2.5). The associated line bundle is $\mathbf{L} = P \times_{S^1} \mathbb{C}$, where S^1 acts on $\mathbb{C}$ by $\lambda : v \mapsto \lambda v$. It is isomorphic to the line bundle constructed in (2.5).

2.3. **All Bott towers arise in this way.**

Definition 2.6. An *isomorphism* between two Bott towers is a collection of holomorphic diffeomorphisms $F_j : M_j \longrightarrow M_j'$ which commute with the maps $\pi_j, \sigma_j^N, \sigma_j^S$ and $\pi_j', \sigma_j^N{}', \sigma_j^S{}'$.

The constructions in §2.2 give a map

$$\mathbb{Z}^{n(n-1)/2} \longrightarrow \{\text{Isomorphism classes of } n\text{-step Bott towers}\} \tag{2.7}$$

and – given an M_n in the image of (2.7) – a map

$$\mathbb{Z}^n \longrightarrow \{\text{Isomorphism classes of holomorphic line bundles over } M_n\}. \tag{2.8}$$

In this section we will show that these two maps are *bijections.*

For every $1 \leq k \leq n$ we have a $\mathbb{C}P^1 \hookrightarrow M_n$, which we think of as the kth step of the tower. We define it in the following way. Given a subset $Q \subseteq M_{i-1}$ we construct subsets $C_F(Q), C_S(Q) \subseteq M_i$ by $C_F(Q) = \pi_i^{-1}(Q)$ and $C_S(Q) = \sigma_i^S(Q)$. Let the sequence $[A_1, \ldots, A_n]$, where $A_i \in \{S, F\}$, denote the subset of M_n which is given by $C_{A_n} \cdots C_{A_2} C_{A_1}(M_0)$. In particular, define $S_k^{(n)} = [A_1, \ldots, A_n]$ where $A_i = S$ for all $i \neq k$ and $A_k = F$. This is a $\mathbb{C}P^1$ embedded in M_n.

Define a map $H^2(M_n, \mathbb{Z}) \longrightarrow \mathbb{Z}^n$ by

$$\alpha \mapsto \alpha(S_1^{(n)}), \ldots, \alpha(S_n^{(n)}). \tag{2.9}$$

Remark 2.10. The definition of the map (2.9) involves the choice of an orientation of the $S_j^{(n)}$. Each of these is naturally isomorphic to $\mathbb{C}P^1$. We choose the orientation as follows. Restrict to the open dense set $\{[1,w]\} \subseteq \mathbb{C}P^1$ and write $w = x + iy$, then $\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\}$ is an oriented basis. We choose this convention so that the integral of the curvature of the tautological line bundle equals to -1. Similarly, on a complex manifold with local coordinates $w_1, \ldots, w_n$ we take $\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial x_n}\}$ to be an oriented basis.

Lemma 2.11. *The map (2.9) gives an isomorphism*

$$H^2(M_n, \mathbb{Z}) \cong \mathbb{Z}^n.$$

Proof. We use induction. Assume that $H^1(M_{n-1}, \mathbb{Z}) = H^3(M_{n-1}, \mathbb{Z}) = \{0\}$ and that the map

$$\alpha \mapsto \alpha(S_1^{(n-1)}), \ldots, \alpha(S_{n-1}^{(n-1)}) \tag{2.12}$$

induces an isomorphism $H^2(M_{n-1}, \mathbb{Z}) \cong \mathbb{Z}^{n-1}$. Consider the Leray-Serre spectral sequence that corresponds to the fibration $M_n \longrightarrow M_{n-1}$. Its E_2 term is

2	$\mathbb{Z}$	0	$\mathbb{Z}^{n-1}$	0	
1	0	0	0	0	$\cdots$
0	$\mathbb{Z}$	0	$\mathbb{Z}^{n-1}$	0	
	0	1	2	3	

Thus we have $H^1(M_n,\mathbb{Z})=0$, $H^3(M_n,\mathbb{Z})=0$, and an exact sequence

$$0 \longrightarrow \mathbb{Z}^{n-1} \longrightarrow H^2(M_n,\mathbb{Z}) \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0.$$

The map π is given by integration over the fiber, which is the set $S_n^{(n)}$. We can write

$$H^2(M_n,\mathbb{Z}) \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z} \tag{2.13}$$

where the $\mathbb{Z}^{n-1}$ term is given by restriction to the South pole section, $\sigma_n^S(M_{n-1})$, which is identified with M_{n-1}, and where we identify $H^2(M_{n-1}) \cong \mathbb{Z}^{n-1}$ by (2.12). Since $\sigma_n^S(S_k^{(n-1)}) = S_k^{(n)}$, the isomorphism (2.13) is given by the map (2.9). □

Lemma 2.14. *Let* $\mathbf{L} \longrightarrow M_n$ *be a holomorphic line bundle. Then up to isomorphism,* $\mathbf{L}$ *is determined by its Chern class.*

Proof. Let $\mathcal{O}$ be the sheaf of holomorphic functions on M_n and let $\mathcal{O}^\times$ be the sheaf of non-vanishing holomorphic functions. Then we have a long exact sequence

$$\ldots \longrightarrow H^1(M_n,\mathcal{O}) \longrightarrow H^1(M_n,\mathcal{O}^\times) \xrightarrow{c_1} H^2(M_n,\mathbb{Z}) \longrightarrow \ldots.$$

The group $H^1(M_n,\mathcal{O}^\times)$ is the Picard group of holomorphic line bundles over M_n and the map c_1 is the Chern class. It is thus sufficient to show that $H^1(M_n,\mathcal{O}) = \{0\}$. We show this by induction on n.

For $n=1$, an easy computation shows that $H^0(\mathbb{C}P^1,\mathcal{O}) = \mathbb{C}$ and $H^i(\mathbb{C}P^1,\mathcal{O}) = 0$ for all $i \geq 1$. (For instance, this is a special case of a computation done in §2.7).

For a general n, consider the fibration $M_n \longrightarrow M_{n-1}$. The cohomology of the total space M_n can be computed by Leray's spectral sequence.

Thus the E_2 term in the spectral sequence has just one row;

	$\vdots$	$\vdots$	$\vdots$	
1	0	0	0	$\cdots$
0	$H^0(M_{n-1},\mathcal{O})$	$H^1(M_{n-1},\mathcal{O})$	$H^2(M_{n-1},\mathcal{O})$	$\cdots$
	0	1	2	

from which we see that $H^*(M_n,\mathcal{O}) = H^*(M_{n-1},\mathcal{O})$. By induction, this cohomology is trivial. □

Lemma 2.15. *The map (2.8) is a bijection.*

Proof. Let M_n be a Bott tower of the form (2.2). By Lemma 2.14, an inverse map to (2.8) is given by the Chern class

$$\{\text{line bundles}\} \xrightarrow{c_1} H^2(M_n,\mathbb{Z})$$

followed by the isomorphism $H^2(M_n,\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}^n$ given in Lemma 2.11. □

Lemma 2.16. *The map (2.7) is a bijection.*

Proof. We first show that (2.7) is onto. Let M_n be a Bott tower, then $M_n = \mathbb{P}(\mathbf{1} \oplus \mathbf{L_n})$ where $\mathbf{L}_n$ is a holomorphic line bundle over an $(n-1)$-step Bott tower, M_{n-1}. By induction, we assume that M_{n-1} is of the form (2.2). By Lemma 2.15, $\mathbf{L}_n$ is isomorphic to a line bundle as constructed in (2.4), (2.5). Thus M_n is of the form (2.2), (2.3) with $c_{1,n} = l_1, \ldots, c_{n-1,n} = l_{n-1}$.

We now show that (2.7) is one to one. For this, it is sufficient to reconstruct the line bundle $\mathbf{L}_j$ from the data $\pi_j : M_j \longrightarrow M_{j-1}$ and $\sigma_j^N, \sigma_j^S : M_{j-1} \longrightarrow M_j$. For $p \in M_{j-1}$, let Q_p be the tautological bundle over $\pi_j^{-1}(p)$. The Q_p fit together to form a line bundle $Q \longrightarrow M_j$. Let $\mathbf{L}_N = (\sigma_j^N)^* Q$ and $\mathbf{L}_S = (\sigma_j^S)^* Q$, then $\mathbf{L}_j = \mathbf{L}_N \otimes \mathbf{L}_S^*$. □

2.4. **Torus actions.** A *torus action* on a Bott tower $\{M_j\}_{j=1}^n$ is a holomorphic action of a torus T on each M_j such that the maps $\pi_j, \sigma_j^N, \sigma_j^S$ are equivariant.

A *complete* torus action on the tower $\{M_j\}_{j=1}^n$ is a torus action with $\dim T = n$ for which the action on M_n is *infinitesimally effective*, i.e., $\{a \in T \mid ap = p \ \ \forall p \in M_n\}$ is a discrete subgroup of T^n.

Example 2.17. Let $k \in \mathbb{Z}$, let S^1 act on $\mathbb{C}P^1$ on the left by $\lambda \cdot [z, w] = [z, \lambda^k w]$. Note that $\mathbb{C}P^1$ is a 2-sphere and the action of S^1 rotates it k times. This is a complete torus action if $k \neq 0$; all complete torus actions on $\mathbb{C}P^1$ arise in this way.

The *standard* torus action on M_n is the action of $T^n = S^1 \times \ldots \times S^1$ given by

$$(\lambda_1, \ldots, \lambda_n) \cdot [z_1, w_1, \ldots, z_n, w_n] = [z_1, \lambda_1 w_1, \ldots, z_n, \lambda_n w_n] \tag{2.18}$$

in the notation of §2.2. We get other torus actions by composing this with any homomorphism $T \longrightarrow T^n$.

Lemma 2.19. *All torus actions are obtained in this way.*

Proof. It is sufficient to show that the elements of $(\mathbb{C}^\times)^n$ acting by (2.18) exhaust all the automorphisms of M_n. For $n = 1$, the complex automorphisms of $\mathbb{C}P^1$ which fix the north and south poles are given by the action of $\mathbb{C}^\times$ as in Example 2.17. Thus if $F : \{M_j\}_{j=1}^n \longrightarrow \{M_j\}_{j=1}^n$ is an automorphism which is trivial on M_{n-1} then on the fiber over $p \in M_{n-1}$ it acts as in Example 2.17 with $\lambda = \lambda(p)$. The function $\lambda : M_{n-1} \longrightarrow \mathbb{C}$ is holomorphic, therefore it is constant. The rest follows by induction. □

Fix a holomorphic line bundle $\pi_{\mathbf{L}} : \mathbf{L} \longrightarrow M_n$. A *torus action on* $\mathbf{L}$ is a holomorphic action of a torus T on $\mathbf{L}$ and on each M_j such that the maps π_j, σ_j^N, σ_j^S and $\pi_{\mathbf{L}}$ are equivariant. A *complete torus action* on $\mathbf{L}$ is an action of T which is infinitesimally effective on $\mathbf{L}$ and where $\dim T = n+1$. The extra S^1 in this action is added for technical reasons. This will make an embedding in § 3.6 more natural.

The *standard* torus action on $\mathbf{L}$ is the action of $T^{n+1} = T^n \times S^1$ given by

$$(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) \cdot [z_1, \ldots, w_n, v] = [z_1, \lambda_1 w_1, \ldots, z_n, \lambda_n w_n, \lambda_{n+1} v] \tag{2.20}$$

in the notation of §2.2. Every torus action on $\mathbf{L}$ is given by a homomorphism $T \longrightarrow T^{n+1}$ composed with the action (2.20).

Suppose that we have a principal S^1-bundle $P \longrightarrow M_n$ as described in §2.2. Then we define torus actions, complete torus actions and the standard torus action on P in the same way as above.

2.5. **Twisted cubes.** Take an affine space V with a lattice ℓ_V. A *twisted cube* C in V consists of a subset of V, called the *support* of C, and a *density function* $\rho : V \longrightarrow \mathbb{R}$ which takes the values 1 and -1 on the support. This data will determine a signed measure m_C in V.

A twisted cube in $V = \mathbb{R}^2$ can look like:

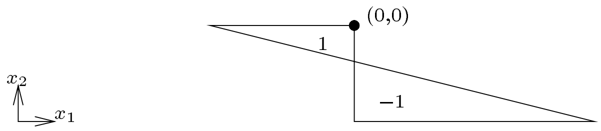

FIGURE 1. $n = 2$, Twisted cube

Combinatorially it resembles a cube, but its faces may have various angles and the intersection of faces may not be a face.

To a collection of integers $\{c_{ij}\}_{1\le i<j\le n}$ and real numbers $l_1, \dots, l_n$ we associate a *standard twisted cube* C_0 in the following way. We set $V = \mathbb{R}^n$ and $\ell_V = \mathbb{Z}^n$. The support of C_0 is defined to be the set of all $(x_1, \dots, x_n) \in \mathbb{R}^n$ which satisfy

$$\begin{aligned} -l_n \le x_n \le 0 \quad &\text{or} \quad 0 < x_n < -l_n \\ -(l_{n-1} + c_{n-1,n}x_n) \le x_{n-1} \le 0 \quad &\text{or} \quad 0 < x_{n-1} < -(l_{n-1} + c_{n-1,n}x_n) \\ &\vdots \\ -(l_1 + c_{12}x_2 + \dots) \le x_1 \le 0 \quad &\text{or} \quad 0 < x_1 < -(l_1 + c_{12}x_2 + \dots) \end{aligned} \tag{2.21}$$

Define $\operatorname{sign}^-(x) = 1$ for $x > 0$ and $\operatorname{sign}^-(x) = -1$ for $x \le 0$. The density function is then defined to be $\rho_0(x) = (-1)^n \operatorname{sign}^-(x_1) \cdots \operatorname{sign}^-(x_n)$ on this set and 0 elsewhere. Note that in the above picture we had $l_1, l_2 > 0$ and $c_{12} > 0$.

The signed measure m_{C_0} is defined to be the function $\rho_0(x)$ times Lebesgue measure. The reason that some inequalities in (2.21) are strict and others are not will be made clear in §2.7.

Definition 2.22. A *twisted cube* C in (V, ℓ_V) is constructed from a standard twisted cube C_0 in $\mathbb{R}^n$ and from an affine isomorphism $A : \mathbb{R}^n \longrightarrow V$ which sends $\mathbb{Z}^n$ onto ℓ_V. We then define $\rho(Ax) = \rho_0(x)$ and $m_C = \rho(\alpha)|d\alpha|$ where $|d\alpha|$ is Lebesgue measure on V, normalized so that V/ℓ_V has volume 1.

2.6. **The Duistermaat-Heckman measure.**

Theorem 2. *Let M_n be a Bott-tower, let T act on it by a complete torus action and let ω be a closed T-invariant 2-form. Let $\Phi : M_n \longrightarrow \mathfrak{t}^*$ be a moment map. Then the corresponding D-H measure coincides with the measure m_C for a twisted cube in $\mathfrak{t}^*$.*

Proof. of Theorem 2. Let $(l_1, \dots, l_n)$ be the image of ω under the isomorphism $H^2(M_n, \mathbb{R}) \cong \mathbb{R}^n$ which comes from Lemma 2.11. By Lemma 2.16 we can assume that M_n is given by our standard construction of §2.2. Let $(c_{ij})_{1\le i<j\le n}$ be the corresponding integers, as in (2.7).

By Lemma 2.19, Remark 1.4 and Definition 2.22 it is sufficient to prove the theorem for the standard action of $T^n = S^1 \times \dots \times S^1$. Note that if T acts effectively then the lattice $\ell_V \subset i\mathfrak{t}^* \cong \mathfrak{t}^*$ is the integral weight lattice. Otherwise it is an appropriate sublattice.

By Theorem 1, it is sufficient to prove the theorem for *any* 2-form ω' in the same cohomology class as ω. Let C_0 be the standard twisted cube in $\mathbb{R}^n$ which is determined by $\{c_{ij}\}$ and $\{l_j\}$. We will find an ω' for which we can compute the D-H measure explicitly and it will turn out to be the measure m_{C_0}.

Recall that $M_n = (S^3)^n/K$, where the ith factor of $K = (S^1)^n$ acts by

$$(z_1, \dots, w_n) \cdot a_i = (z_1, \dots, w_{i-1}, z_i a_i, w_i a_i, \dots, z_j, a_i^{c_{ij}} w_j, \dots). \tag{2.23}$$

If we write $z_j = |z_j| e^{2\pi i \alpha_j}$ and $w_j = |w_j| e^{2\pi i \beta_j}$, then the action (2.23) is generated by the vector fields

$$\eta_i = \frac{\partial}{\partial \alpha_i} + \frac{\partial}{\partial \beta_i} + \sum_{j=i+1}^{n} c_{i,j} \frac{\partial}{\partial \beta_j}.$$

The standard torus action is generated by the vector fields

$$\xi_i = \frac{\partial}{\partial \beta_i}.$$

Define

$$h_i = |w_i|^2.$$

We think of $h_i : M_n \longrightarrow [0,1]$ as the height function on the $\mathbb{C}P^1$ in the ith step of the tower. The common level sets of $(h_1, \dots, h_n) : M_n \longrightarrow [0,1]^n$ are exactly the T-orbits in M_n; this identifies M_n/T^n with a hypercube.

We will now show there is a closed invariant two form ω', such that $[\omega'] = [\omega]$ and whose pull-back to $(S^3)^n$ is

$$\tilde{\omega} = -d\theta$$

where θ is an invariant 1-form given by

$$\theta = \sum_{j=1}^{n} f_j \left((1 - h_j) d\alpha_j + h_j d\beta_j \right)$$

for some functions f_i. For $\tilde{\omega}$ to descend to M_n, we require $\iota(\eta_i)\tilde{\omega}$ to vanish for all i. Note that

$$\iota(\eta_i)\tilde{\omega} = d\iota(\eta_i)\theta = d(f_i + \sum_{j=i+1}^{n} c_{ij} f_j h_j)$$

assuming the invariance of θ. This leads to an inductive definition of the functions f_i. We set $f_n = l_n$ and, given $\{f_j \ ; \ j > i\}$, we set

$$f_i = l_i - \sum_{j=i+1}^{n} c_{ij} f_j h_j. \tag{2.24}$$

With this choice, $\tilde{\omega}$ descends to a closed, invariant 2-form ω' on M_n. We now show that ω' has the same cohomology class as ω.

Fix j and consider the map of $[0,1] \times [0,1]$ to M_n given by sending (p_j, q_j) to the point $[z_1, \dots, w_n]$ with $(z_j, w_j) = (1, \sqrt{p_j} e^{2\pi i q_j})$ and $(z_i, w_i) = (1, 0)$ for all other i. The image is the sphere $S_j^{(n)}$ defined in §2.3 and h_j, β_j form cylindrical coordinates on this sphere. The pull-back of θ is $l_j h_j d\beta_j$, the pull-back of ω' is $l_j d\beta_j \wedge dh_j$. Its integral is l_j (using the convention in Remark 2.10). Thus we see that $\int_{S_j^{(n)}} \omega' = l_j = \int_{S_j^{(n)}} \omega$ for all j, i.e., that ω' and ω are cohomologous.

One can easily verify that

$$\Phi_j = -\iota(\xi_j)\theta = -f_j h_j \quad \text{for} \quad j = 1, \dots, n \tag{2.25}$$

define a moment map $\Phi = (\Phi_1, \dots, \Phi_n) : M_n \longrightarrow \mathbb{R}^n$.

Once we fix the values of $h_{i+1}, \dots, h_n$, and thus the value of f_i by (2.24), the image of Φ_i is by (2.25) either $\{-f_i \leq x_i \leq 0\}$ or $\{0 \leq x_i \leq -f_i\}$, according to the sign of f_i. Substituting (2.24) and $x_j = \Phi_j = -f_j h_j$, we get that the image of Φ is the closure of the twisted cube (2.21).

We now compute Liouville measure and its push-forward. The set $\{\alpha_i = 0 \textit{ for all } i\} \subset S^3$ maps surjectively onto M. On this set we have $d\alpha_i \equiv 0$, and the functions (h_i, β_i) are local coordinates. Thus $\theta = \sum f_j h_j d\beta_j = -\sum \Phi_j d\beta_j$, so $\omega' = \sum d\Phi_j \wedge d\beta_j$ and

$$\omega^n/n! = d\Phi_1 \wedge d\beta_1 \wedge \ldots \wedge d\Phi_n \wedge d\beta_n. \tag{2.26}$$

Pushing forwards amounts to integrating over the β_i variables. This gives ± 1 because the β_i's range over the interval $[0,1]$. Since $|d\Phi_1 \wedge \ldots \wedge d\Phi_n|$ is Lebesgue measure on $\mathfrak{t}^*$, we have $\Phi_* \omega^n/n! = \pm$(*Lebesgue measure on* image(Φ)), where the sign depends on whether or not ω^n is compatible with the orientation on M. This orientation comes from the complex structure on M as in Remark 2.10. We restrict to the set $w_i \neq 0$, $\alpha_i \equiv 0$, $i = 1, \dots, n$ and take w_i, $i = 1, \dots, n$ as complex coordinates on this set. The corresponding orientation is given by the volume form

$$d\beta_1 \wedge dh_1 \wedge \ldots d\beta_n \wedge dh_n. \tag{2.27}$$

By (2.25) and (2.24) we have $d\Phi_i = -f_i dh_i - h_i d(l_i - \sum_{j>i} c_{ij} f_j h_j) = -f_i dh_i + h_i \sum_{j>i} e_{ij} dh_j$ for some constants e_{ij}. Therefore we have $d\Phi_1 \wedge \ldots \wedge d\Phi_n = (-1)^n f_1 \cdots f_n \; dh_1 \wedge \ldots \wedge dh_n$. This implies that ω^n (2.26) is compatible with the orientation (2.27) if and only if $f_1 \cdots f_n > 0$. Finally, since sign$(f_1 \cdots f_n) = (-1)^n$ sign$(\Phi_1 \cdots \Phi_n)$ by (2.25) and setting $x_i = \Phi_i$, we get that $\Phi_* \omega^n/n! = (-1)^n$sign$(x_1 \cdots x_n) \cdot$ (*Lebesgue measure on image*(Φ)), which is exactly m_{C_0} by §2.5. □

2.7. **The virtual character.** Let $\pi : \mathbf{L} \longrightarrow M$ be a holomorphic line bundle over a complex manifold with an action of a torus T. Let $\mathcal{O}_{\mathbf{L}}$ be the sheaf of holomorphic sections. The *equivariant index* is the formal sum of representations of T

$$\textit{index}(M, \mathcal{O}_{\mathbf{L}}) = \sum (-1)^i H^i(M, \mathcal{O}_{\mathbf{L}}).$$

The *virtual character* is the function $\mathcal{X} : T \longrightarrow \mathbb{C}$ which is given by $\mathcal{X} = \sum (-1)^i \mathcal{X}^i$ where $\mathcal{X}^i(a) = \text{trace}\{a : H^i(M, \mathcal{O}_{\mathbf{L}}) \longrightarrow H^i(M, \mathcal{O}_{\mathbf{L}})\}$ for $a \in T$. Every μ in the integral weight lattice $\ell^* \subset i\mathfrak{t}^*$ defines a homomorphism $e^\mu : T \longrightarrow S^1$. We can write $\mathcal{X}_{M,T} = \sum_{\mu \in \ell^*} m_\mu e^\mu$. The coefficients are given by a function *mult* : $\ell^* \longrightarrow \mathbb{Z}$, sending $\mu \mapsto m_\mu$, called the *multiplicity function* for the index.

Remark 2.28. Let T' act on $\mathbf{L}$ by a homomorphism $A : T' \longrightarrow T$ composed with the action of T. Denote $L = (dA)^* : i\mathfrak{t}^* \longrightarrow i\mathfrak{t}'^*$. Then the multiplicity functions which correspond to the actions of T and T' are related by $\textit{mult}'(\alpha') = \sum_{\alpha \in \ell^* \cap L^{-1}(\alpha')} \textit{mult}(\alpha)$. (Compare with Remark 1.4).

We will now show that if $M = M_n$ is a Bott tower then the multiplicity function is given by the density of a twisted cube. In particular, all the weights occur with a multiplicity -1, 0 or 1.

Theorem 3. *Fix integers $(c_{ij})_{1\le i<j\le n}$ and $(l_j)_{1\le j\le n}$. Let $\mathbf{L} \longrightarrow M_n$ be the corresponding line bundle over a Bott tower as in §2.2. Let C_0 be the twisted cube which is determined by these integers (see §2.5), and let $\rho_0 : \mathbb{R}^n \longrightarrow \{-1,0,1\}$ be its density function. Consider the standard complete torus action of $T = T^n \times S^1$ as in (2.20). Then the multiplicity function for $\ell^* \cong \mathbb{Z}^n \times \mathbb{Z}$ is given by $mult(\alpha,1) = \rho_0(i^{-1}\alpha)$ for all $\alpha \in \ell^*$ and $mult(\alpha,k) = 0$ for all $k \neq 1$.*

For the other complete torus actions, we have:

Corollary 2.29. Take a line bundle over a Bott tower with a complete torus action, as in §2.4. Let $mult : i\mathfrak{t}^* \longrightarrow \mathbb{Z}$ be the corresponding multiplicity function. Then there is a hyperplane V in $\mathfrak{t}^*$ with a lattice $\ell_V \subseteq i\ell^* \cap V$, and a twisted cube C in V with a density function $\rho : i^{-1}V \longrightarrow \{-1,0,1\}$, such that for all $\alpha \in \ell^*$, if $\alpha \notin \ell_V$ then $m(\alpha) = 0$ and if $\alpha \in \ell_V$ then $m(\alpha) = \rho(i^{-1}\alpha)$.

Proof. of Corollary. This follows from Theorem 3 by §2.3, Remark 2.28, and Definition 2.22. □

Proof. of Theorem 3. We first compute the cohomology of $\mathbb{C}^2 \smallsetminus \mathbf{0}$ with coefficients in the sheaf $\mathcal{O}$ of holomorphic functions; see [G-H, §0.2]. Take the covering $\mathfrak{U} = \{U_1, U_2\}$ where $U_1 = \mathbb{C} \times \mathbb{C}^\times$ and $U_2 = \mathbb{C}^\times \times \mathbb{C}$. This covering is *good*, i.e., $H^q(U_1) = H^q(U_2) = H^q(U_1 \cap U_2) = 0$ for $q \ge 0$. The holomorphic functions are

$$\Gamma_{\mathrm{hol}}(U_1) = \{ \sum_{i,j\in\mathbb{Z},\ i\ge 0} a_{ij} z^i w^j \}, \quad \Gamma_{\mathrm{hol}}(U_2) = \{ \sum_{i,j\in\mathbb{Z},\ j\ge 0} a_{ij} z^i w^j \}$$

and

$$\Gamma_{\mathrm{hol}}(U_1 \cap U_2) = \{ \sum_{i,j\in\mathbb{Z}} a_{ij} z^i w^j \}.$$

Consider the map $\Gamma_{\mathrm{hol}}(U_1) \oplus \Gamma_{\mathrm{hol}}(U_2) \xrightarrow{\delta} \Gamma_{\mathrm{hol}}(U_1 \cap U_2)$ given by $(f,g) \mapsto g|_{U_1\cap U_2} - f|_{U_1\cap U_2}$. Recall that $H^0(\mathfrak{U},\mathcal{O}) = \ker\delta$ and $H^1(\mathfrak{U},\mathcal{O}) = \operatorname{coker}\delta$. The torus $T^2 = (S^1)^2$ acts on the holomorphic functions by $((a,b)\cdot f)(z,w) = f(a^{-1}z, b^{-1}w)$. This action descends to the cohomology. The corresponding weight-spaces for the weight $(i,j) \in \mathbb{Z}^2$ are

$$H^0(\mathbb{C}^2 \smallsetminus \mathbf{0})_{(\mathbf{i},\mathbf{j})} = \begin{cases} \mathrm{span}(z^{-i}w^{-j}) & \text{if } i \le 0 \text{ and } j \le 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^1(\mathbb{C}^2 \smallsetminus \mathbf{0})_{(\mathbf{i},\mathbf{j})} = \begin{cases} \mathrm{span}(z^{-i}w^{-j}) & \text{if } i > 0 \text{ and } j > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the multiplicity function is $mult(\alpha) = 1$ for α in the closed negative quadrant, $mult(\alpha) = -1$ for α in the open positive quadrant, and $mult(\alpha) = 0$ otherwise.

We now compute $H^*(M_n, \mathcal{O}_{\mathbf{L}})$. As in §2.2 we write $\mathbf{L} = (\mathbb{C}^2 \smallsetminus \mathbf{0})^{\mathbf{n}} \times_{\mathbf{G}} \mathbb{C}$ where $G = (\mathbb{C}^\times)^n$ and its ith factor acts by (2.5);

$$(z_1, \dots, w_n, v) \cdot a_i = (z_1, \dots, w_{i-1}, z_i a_i, w_i a_i, \dots, z_j, a_i^{c_{ij}} w_j, \dots, a_i^{l_i} v). \tag{2.30}$$

Consider the good covering $\tilde{\mathfrak{U}}$ of $(\mathbb{C}^2 \smallsetminus \mathbf{0})^{\mathbf{n}}$ given by the 2^n sets $U_{i_1} \times \cdots \times U_{i_n}$ for $i_1, \dots, i_n \in \{1,2\}$, where $U_1 = \mathbb{C} \times \mathbb{C}^\times$ and $U_2 = \mathbb{C}^\times \times \mathbb{C}$ as before. This descends to a good covering $\mathfrak{U}$ of M_n; every set in $\mathfrak{U}$ is isomorphic to $\mathbb{C}^n$, with coordinates z_j/w_j or w_j/z_j, and every intersection of sets in $\mathfrak{U}$ is isomorphic to a product of $\mathbb{C}$'s and $\mathbb{C}^\times$'s.

The natural pull-back $H^*(\mathfrak{U}, \mathcal{O}_{\mathbf{L}}) \longrightarrow H^*(\tilde{\mathfrak{U}}, \mathcal{O})$ is an injection onto the G-invariant part of $H^*(\tilde{\mathfrak{U}}, \mathcal{O})$. This follows from the fact that holomorphic sections of $\mathcal{O}_{\mathbf{L}}$ are given by holomorphic sections of $\mathcal{O}$ which are G-invariant with respect to the action (2.30); see [K-T, §8]. Since $\mathfrak{U}$ and $\tilde{\mathfrak{U}}$ are good coverings, it follows that $H^*(M_n, \mathcal{O}_{\mathbf{L}})$ is isomorphic to the G-invariant part of $H^*\left(((\mathbb{C}^2 \smallsetminus \mathbf{0})^{\mathbf{n}}, \mathcal{O})^n\right)$.

We now compute $H^*((\mathbb{C}^2 \smallsetminus \mathbf{0})^{\mathbf{n}}, \mathcal{O})$. Consider the natural action of T^{2n}. The weights are multi-indices $(I, J) \in \mathbb{Z}^n \times \mathbb{Z}^n$; we write $I = (i_1, \dots, i_n)$ and $J = (j_1, \dots, j_n)$. Recall that we defined $\operatorname{sign}^-(j) = 1$ if $j > 0$ and $= -1$ if $j \leq 0$. From the cohomology of $\mathbb{C}^2 \smallsetminus \mathbf{0}$ that we have computed and from the Künneth formula it follows that $H^k(\mathbb{C}^2 \smallsetminus \mathbf{0})_{(\mathbf{I},\mathbf{J})}$ is either zero or that it is one dimensional and spanned by the monomial $z^{-i_1}w^{-j_1} \cdots z^{-i_n}w^{-j_n}$. The latter occurs if for all ℓ we have $\operatorname{sign}^-(i_\ell) = \operatorname{sign}^-(j_\ell) =$, say, ϵ_ℓ; and $\epsilon_\ell = 1$ for exactly k indices among $1, \dots, n$. In particular, $(-1)^k = (-1)^n \operatorname{sign}^-(j_1) \cdots \operatorname{sign}^-(j_n)$.

The action of $G = (\mathbb{C}^\times)^n$ on $(\mathbb{C}^2 \smallsetminus \mathbf{0})^{\mathbf{n}} \times \mathbb{C}$ induces an action on functions by

$$(a_k f)(z_1, \dots, w_n) = a_k^{l_k} f(z_1, \dots, w_{k-1}, z_k a_k^{-1}, w_k a_k^{-1}, \dots, z_\ell, a_k^{-c_{k\ell}} w_\ell, \dots).$$

The monomial $z^{-i_1}w^{-j_1} \cdots z^{-i_n}w^{-j_n}$ is then a G weight vector with a weight in $\mathbb{Z}^n$ whose kth coordinate is $l_k + i_k + j_k + \sum_{\ell=k+1}^n c_{k\ell} j_\ell$. Thus the G-invariant part of $H^*((\mathbb{C}^2 \smallsetminus \mathbf{0})^{\mathbf{n}}, \mathcal{O})$ consists of those monomials $z^{-i_1}w^{-j_1} \cdots z^{-i_n}w^{-j_n}$ for which

$$\begin{aligned} l_1 + i_1 + j_1 + c_{12}j_2 + c_{13}j_3 + \ldots + c_{1n}j_n &= 0 \\ l_2 + i_2 + j_2 + c_{23}j_3 + \ldots + c_{2n}j_n &= 0 \\ &\vdots \\ l_n + i_n + j_n &= 0 \end{aligned} \tag{2.31}$$

The action (2.20) induces a T action on the functions given by

$$((\lambda_1, \dots, \lambda_n, \lambda_{n+1}) \cdot f)(z_1, \dots, w_n) = \lambda_{n+1} f(z_1, \lambda_1^{-1} w_1, \dots, z_n, \lambda_n^{-1} w_n).$$

The weight of the monomial $z^{-i_1}w^{-j_1} \cdots z^{-i_n}w^{-j_n}$ with respect to this T action is $(j_1, \dots, j_n, 1)$. Thus the index of $(M_n, \mathcal{O}_{\mathbf{L}})$ is given by the set of $(x_1, \dots, x_n, 1) = (j_1, \dots, j_n, 1)$ for which there exist $(i_1, \dots, i_n)$ such that (2.31) is satisfied and such that $\operatorname{sign}^-(i_\ell) = \operatorname{sign}^-(j_\ell)$ for all ℓ. This is exactly the set (2.21). The multiplicity is $(-1)^n \operatorname{sign}^-(j_1) \cdots \operatorname{sign}^-(j_n) = (-1)^n \operatorname{sign}^-(x_1) \cdots \operatorname{sign}^-(x_n) = \rho_0(x)$. □

In the following two propositions we give formulas for the virtual character $\boldsymbol{\chi} : T \longrightarrow \mathbb{C}$, for a line bundle over a Bott-tower. Recall that for every integral weight $\mu \in \ell^*$ we have a homomorphism $e^\mu : T \longrightarrow S^1$. Denote by $\mathbb{Z}[T]$ the integral combinations of these e^μ's. Then $\boldsymbol{\chi} \in \mathbb{Z}[T]$ is given by $\boldsymbol{\chi} = \sum_{\mu \in \ell^*} m_\mu e^\mu$ where $m_\mu = mult(\mu)$ was computed in Theorem 3.

In particular, for the standard torus $T^{n+1} = S^1 \times \ldots \times S^1$ we have a natural identification $i(\mathfrak{t}^{n+1})^* = \mathbb{R}^{n+1}$ with $\mathbb{Z}^{n+1}$ being the integral weight lattice, so that $e^{\boldsymbol{\mu}} : \boldsymbol{a} \mapsto a_1^{\mu_1} \cdots a_{n+1}^{\mu_{n+1}}$.

Proposition 2.32. *Consider the standard torus action of T^{n+1} on $\mathbf{L} \longrightarrow M_n$. Denote the standard basis in $\mathbb{R}^{n+1}$ by $f_1, \dots, f_{n+1}$. Then the equivariant index is given by the following element of $\mathbb{Z}[T^{n+1}]$;*

$$\boldsymbol{\chi} = D_1 \cdots D_n(e^{f_{n+1}})$$

when the operators $D_i : \mathbb{Z}[T^{n+1}] \longrightarrow \mathbb{Z}[T^{n+1}]$ are defined using c_{ij}, l_j in the following way.

$$D_i(e^{\boldsymbol{\mu}}) = \begin{cases} e^{\boldsymbol{\mu}} + e^{(\boldsymbol{\mu}-f_i)} + \ldots + e^{(\boldsymbol{\mu}-k_i f_i)} & \text{if } k_i \geq 0 \\ 0 & \text{if } k_i = -1 \\ -e^{(\boldsymbol{\mu}+f_i)} - \ldots - e^{(\boldsymbol{\mu}-(k_i+1)f_i)} & \text{if } k_i \leq -2 \end{cases}$$

and the functions k_i are defined as follows; if $\boldsymbol{\mu} = \sum x_i f_i$ then $k_i(\boldsymbol{\mu}) = l_i + \sum_{j=i+1}^{n} c_{ij} x_j$.

And for the other torus actions:

Proposition 2.33. *Take a complete torus action of T on $\mathbf{L} \longrightarrow M_n$ which is given by a homomorphism $A : T \longrightarrow T^{n+1}$ followed by the standard action as in §2.4. Let $\beta_i = A^* f_i$ for all i where $\{f_i\}$ are the standard basis elements in $\mathbb{R}^{n+1}$. Then the virtual character is given by*

$$\chi = \tilde{D}_1 \cdots \tilde{D}_n(e^{\beta_{n+1}})$$

where the operators $\tilde{D}_i : \mathbb{Z}[T] \longrightarrow \mathbb{Z}[T]$ are defined by

$$\tilde{D}_i(e^{\boldsymbol{\mu}}) = \begin{cases} e^{\boldsymbol{\mu}} + e^{\boldsymbol{\mu}-\beta_i} + \ldots + e^{\boldsymbol{\mu}-k_i\beta_i} & \text{if } k_i \geq 0 \\ 0 & \text{if } k_i = -1 \\ -e^{\boldsymbol{\mu}+\beta_i} - \ldots - e^{\boldsymbol{\mu}-(k_i+1)\beta_i} & \text{if } k_i \leq -2 \end{cases}$$

and where k_i are defined as follows; if $\boldsymbol{\mu} = \sum x_i \beta_i$ then $k_i(\boldsymbol{\mu}) = l_i + \sum_{j=i+1}^{n} c_{ij} x_j$.

Proof. Both propositions follow immediately from Theorem 3 and Remark 2.28. □

2.8. **Another view of the index.** Whereas Theorem 3 was stated and proved using sheaf cohomology, we could have used Dolbeault cohomology. For this we must choose a connection $\nabla_{\mathbf{L}}$ on the line bundle, a Hermitian structure on $\mathbf{L}$ and a metric on M_n, which are all equivariant. We then take the equivariant index of the twisted Dolbeault operator; $D_{\mathbf{L}} := (\overline{\partial} + \overline{\partial}^*) \otimes \mathbf{1} + \mathbf{1} \otimes \nabla''_{\mathbf{L}}$ where $\nabla''_{\mathbf{L}}$ denotes the anti-holomorphic part of $\nabla_{\mathbf{L}}$. The index will not depend on the choice of connection or metric. We now outline an alternative proof for Theorem 3.

Consider a complete torus action of T on M_n and a lifting of this action to the line bundle $\mathbf{L}$. Let $F \cong \mathbb{C}P^1$ be a fiber of $M_n \longrightarrow M_{n-1}$. We can split the torus into $T = T^{n-1} \times S^1$, where S^1 fixes M_{n-1} and rotates the fibers. We can restrict the Dolbeault operator to the fibers and take the family index; this is the virtual bundle over M_{n-1} denoted $index(D_{\mathbf{L}}|_F)$.

The index over a fiber can be computed using any number of methods. For instance, we can explicitly compute the sheaf cohomology, which is equal to the Dolbeault cohomology by the generalized De Rham theorem; see [K-T, Example 9.1]. Or we could use the generalized Lefschetz formula of Atiyah and Bott, or we could compute the Dolbeault cohomology using Serre duality. This computation for $\mathbb{C}P^1$ tells us that $H^k(F_p, \mathbf{L}|_{F_p})$ must vanish for either $k = 0$ or $k = 1$. Thus in K-theory, $index(D_{\mathbf{L}}|_F)$ is either an honest vector bundle or the negative of an honest bundle. Moreover, this bundle splits into one dimensional pieces under the S^1-action.

Now we use the fact that the index of an equivariant elliptic operator is functorial with respect to pushing forward (families index). This functoriality gives

$$index_T(D_{\mathbf{L}}) = index_{T^{n-1} \times S^1}\left(D_{index_{S^1}(D_{\mathbf{L}}|_F)}\right)$$

which establishes the result by induction.

This was the proof originally given in the first author's thesis [G]. It has the advantage that it only requires an *almost complex* structure to form $D_{\mathbf{L}}$.

Our computation of the index in §2.7 is due to Susan Tolman. It has the advantage of being more elementary and yielding the sharper result, that each individual $H^k(M_{n-1}, \mathcal{O}_L)$ is multiplicity free. This approach was generalized in [K-T] to a computation of the equivariant index of a line bundle over any toric variety. The multiplicities form shapes in $i\mathfrak{t}^*$ which we call *twisted polytopes*.

2.9. **Relation between multiplicities and the D-H measure.** Take a complete action of $T = T^{n+1}$ on a line bundle $\mathbf{L}$ over a Bott tower M_n. It determines an exact sequence

$$0 \longrightarrow S^1 \hookrightarrow T \longrightarrow \tilde{T} \longrightarrow 0 \tag{2.34}$$

where S^1 acts trivially on M_n and rotates the fibers of $\mathbf{L}$, and $\tilde{T}$ acts on M_n by a complete torus action.

The multiplicity function for the equivariant index is then given by a twisted cube C in an affine hyperplane V in $i\mathfrak{t}^*$, by Theorem 3 and Corollary 2.29.

Choose any splitting $\tilde{T} \longrightarrow T$ of (2.34), i.e., a lifting of the $\tilde{T}$-action to $\mathbf{L}$. This determines an affine isomorphism of V with $i\tilde{\mathfrak{t}}^*$. We can write $\mathbf{L} = P \times_{S^1} \mathbb{C}$ where P is a principal S^1-bundle on which $\tilde{T}$ acts. Let ω be the curvature of any invariant connection and let $\Phi : M_n \longrightarrow \mathfrak{t}^*$ be the corresponding moment map, as constructed in §1.4. Then the corresponding D-H measure is the measure m_C which corresponds to the same twisted cube C in $V \cong i\tilde{\mathfrak{t}}^*$.

In particular, the multiplicity function for the index is then equal on the nose to the density function for the D-H measure.

More generally, let M be a compact complex manifold of complex dimension n. Let $\mathbf{L} \longrightarrow M$ be a holomorphic line bundle with a connection whose curvature ω is Kähler. Let T be a k-dimensional torus that acts on M holomorphically, lifts to $\mathbf{L}$ and preserves the connection. On one hand we can form the equivariant index $\sum(-1)^i H^i(M, \mathcal{O}_L)$ and we can consider its multiplicity function $mult : \ell^* \longrightarrow \mathbb{Z}$. On the other hand, we can take the moment map as in (1.2) and let $\rho : \mathfrak{t}^* \longrightarrow \mathbb{R}$ be the density function for the D-H measure. By the Duistermaat-Heckman theorem, ρ is piecewise polynomial and the degrees of the polynomials are at most $n - k$.

The function ρ can then be viewed as a continuous approximation to *mult*. This was shown by Heckman [He] when M is a flag variety, as we described in §1.6. In that setting, H^0 is the representation R_λ and $H^i = 0$ for $i \geq 1$. The relation (1.5) was generalized to arbitrary Kähler manifolds by Guillemin and Sternberg [G-S 2].

This asymptotic relation between *mult* and ρ remains true if ω is not Kähler. This follows from the fixed point formulas of Atiyah and Bott on one hand and of Guillemin, Lerman and Sternberg on the other hand. This relation holds in an even greater generality; for almost complex structures and for *Spin*^{c} structures on M; see [G-K]. Moreover, the localization formulas imply that *mult* and ρ are independent of the respective choices of a holomorphic structure and of a connection; we only need to know the orientation of M and the Chern class of $\mathbf{L}$. (Also see Theorem 1).

In the special case that $\dim T = \frac{1}{2}\dim M$, the functions *mult* and ρ are equal on the nose. As a corollary, one can express the equivariant index in terms of the topological degrees for a map from M/T to $\mathbf{t}^*$ which is induced from the moment map [G-K].

3. Connection with Bott-Samelson manifolds

3.1. **Bott-Samelson manifolds.** Let $\Sigma = \{\alpha_1, \dots, \alpha_r\}$ be an ordering of the simple roots. Let $I = \{i_1, \dots, i_n\} \in \{1, \dots, r\}^n$ be a sequence of natural numbers. Let P_i be the parabolic subgroup associated with α_i; its Lie algebra is $\mathfrak{g}_{\alpha_i} \oplus \mathfrak{b}$. Define

$$\mathbf{P}_I = P_{i_1} \times \cdots \times P_{i_n}.$$

Define a right action of B^n on $\mathbf{P}_I$ as follows;

$$(p_1, \dots, p_n) \cdot \mathbf{b} = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n) \quad \text{for} \quad \mathbf{b} = (b_1, \dots b_n). \tag{3.1}$$

The Bott-Samelson manifold is defined to be the quotient, $M_I = \mathbf{P}_I / B^n$.

The multiplication map $(p_1, \dots, p_n) \mapsto p_1 \cdots p_n$ descends to a well defined map

$$\eta : M_I \longrightarrow G/B.$$

Let P_{i_1} act on $\mathbf{P}_I$ by left multiplication on the first factor, this descends to an action on M_I. The map η is equivariant with respect to the left actions of P_{i_1} on M_I and on G/B.

3.2. **Line bundles over Bott-Samelson manifolds.** Let $e^{\lambda_1}, \dots, e^{\lambda_n}$ be a sequence of weights of the Cartan subgroup H. As in §1.2, we can extend these to homomorphisms $e^{\lambda_i} : B \longrightarrow \mathbb{C}^\times$. Define a representation of B^n on $\mathbb{C}$ by

$$\mathbf{b} \cdot a = e^{\lambda_1}(b_1) \cdots e^{\lambda_n}(b_n) a.$$

Denote this B^n-module by $\mathbb{C}_{\lambda_1, \dots, \lambda_n}$. From this we can build a line bundle over M_I by

$$\mathbf{L}_{I, \lambda_1, \dots, \lambda_n} = \mathbf{P}_I \times_{B^n} \mathbb{C}_{\lambda_1, \dots, \lambda_n}$$

on which P_{i_1} acts on the left.

As in §1.2, given a weight e^λ we have a line bundle $\mathbf{E}_\lambda = G \times_B \mathbb{C}_\lambda$ over G/B. Denote $\mathbf{L}_{I,0,\dots,0,\lambda} = \mathbf{L}_{I,\lambda}$, then we have a commutative diagram of P_{i_1}-equivariant maps;

$$\begin{array}{ccc} \mathbf{L}_{I,\lambda} & \xrightarrow{\tilde{\eta}} & \mathbf{E}_\lambda \\ \downarrow & & \downarrow \\ M_I & \xrightarrow{\eta} & G/B. \end{array} \tag{3.2}$$

3.3. **Connection with Schubert varieties.** The Weyl group W is generated by the simple reflections s_i which correspond to the simple roots α_i. Thus any $w \in W$ has an expression $w = s_{i_1} \cdots s_{i_n}$ of minimal length n. We call such an expression reduced. Let us fix w and fix this expression and consider $I = (i_1, \dots, i_n)$ and the associated Bott-Samelson variety, M_I. The image of the multiplication map η turns out to be the Schubert variety $X_w = \mathit{closure}(B\dot{w}B/B)$, so we have

$$\eta : M_I \longrightarrow X_w. \tag{3.3}$$

One can show that η is an isomorphism between open dense subsets of M_I and X_w, [J, §13.5]. If w has maximal length in W then the image is $X_w = G/B$.

Now consider the parabolic $P = P_J$ which is associated to a subset of the simple roots $J \subseteq \Sigma$ as described in section 1. Let $w \in W^J$ as in §1.1 and let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression, then the composition

$$M_I \longrightarrow G/B \longrightarrow G/P$$

maps M_I onto a Schubert variety in G/P.

The initial importance of (3.3) is that M_I provides a smooth model for the Schubert cell X_w in the following sense; the map (3.2) induces a natural injection

$$\eta^* : \Gamma_{\text{hol}}(X_w, i^*\mathbf{E}_\lambda) \hookrightarrow \Gamma_{\text{hol}}(M_I, \mathbf{L}_{I,\lambda}).$$

The maps $\eta, \tilde{\eta}$ are equivariant with respect to the left actions of B on M_I (by restriction from P_{i_1}) and on X_w. Therefore, η^* implements a monomorphism of B-modules. In fact, η^* turns out to be an isomorphism. Much more is true; Demazure first announced that for all i

$$H^i(X_w, i^*\mathbf{E}_\lambda) \cong H^i(M_I, \mathbf{L}_{I,\lambda}) \tag{3.4}$$

as B-modules. His proof had a serious error which was not found for many years. This was later proved in papers by Mehta, Ramanathan, Seshadri, Ramanan and Andersen. For references see [J, §14, p.395].

We remark that if λ is dominant then $H^i = 0$ for all $i \geq 1$. Let $\rho = \sum_{\alpha \in \Delta^+} \frac{1}{2}\alpha$. For λ not dominant, the Borel-Weil-Bott theorem similarly gives a vanishing of all the cohomologies except for one i depending on the chamber which contains $\lambda + \rho$ [B][J].

3.4. **A family of complex structures.** Recall that $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$. The projection $\mathfrak{b} \rightarrow \mathfrak{h}$ induces a homomorphism of Lie groups $\Upsilon = \Upsilon_\infty : B \longrightarrow H$ as in §1.2. This extends to a whole family of homomorphisms as follows. For $h \in H$ we define $\Upsilon_h : B \longrightarrow B$ by $\Upsilon_h(b) = hbh^{-1}$. We now choose once and for all an $\underline{h} \in i\mathfrak{t}^*$ such that $\alpha(\underline{h}) > 0$ for all $\alpha \in \Delta^+$. We define $\Upsilon_t := \Upsilon_{\exp(t\underline{h})}$.

Proposition 3.5. $\Upsilon_\infty = \lim_{t \longrightarrow \infty} \Upsilon_t$.

Proof. Consider the derivatives of these maps at the identity, $d\Upsilon_t : \mathfrak{b} \longrightarrow \mathfrak{b}$, for $t \leq \infty$. It is sufficient to show that $d\Upsilon_\infty = \lim d\Upsilon_t$. We evaluate these at $\underline{v} \in \mathfrak{b}$. If $\underline{v} \in \mathfrak{h}$ then, since H is commutative, $d\Upsilon_t(\underline{v}) = \underline{v}$ for $t \leq \infty$. If $\underline{v} \in \mathfrak{g}_{-\alpha}$ then $d\Upsilon_t(\underline{v}) = e^{-t\alpha(\underline{h})}\underline{v}$. By assumption, $\alpha(\underline{h}) > 0$, so $\lim_{t \longrightarrow \infty} d\Upsilon_t(\underline{v}) = 0 = d\Upsilon_\infty(\underline{v})$. □

We use the homomorphisms $\Upsilon_t : B \longrightarrow B$ to construct a family of almost complex structures on the Bott-Samelson variety, $M_I = \mathbf{P}_I/B^n$. For $t \leq \infty$, we define a right action R_t of B^n on $\mathbf{P}_I$ by

$$(p_1, \ldots, p_n) \cdot (b_1, \cdots, b_n) = (p_1 b_1, \Upsilon_t(b_1)^{-1} p_2 b_2, \ldots, \Upsilon_t(b_{n-1})^{-1} p_n b_n) \tag{3.6}$$

where as before $(p_1, \ldots, p_n) \in \mathbf{P}_I$ and $(b_1, \ldots, b_n) \in B^n$. Then R_0 coincides with the right action (3.1). Again we quotient $\mathbf{P}_I$ by the right action R_t to get a family of manifolds M_I^t. We denote

$$M_I^t = P_{i_1} \times_{B_t} \ldots \times_{B_t} P_{i_n}/B.$$

The bundles $\mathbf{L}_{I,\lambda_1,\ldots,\lambda_n}$ defined in §3.2 are defined and holomorphic over each M_I^t.

Proposition 3.7. *Fix a sequence* $I = \{i_1, \ldots, i_n\}$ *and weights* $\lambda_1, \ldots, \lambda_n$. *Then the manifolds* M_I^t *are all diffeomorphic, and the bundles* $\mathbf{L}^t_{I,\lambda_1,\ldots,\lambda_n}$ *are smoothly isomorphic, for* $t \leq \infty$.

Proof. Let K_i be the maximal compact subgroup of P_i. Recall that $K_i \cap B = T$. Define a right action of $T^{(n)} := T \times T \times \cdots \times T$ on $K_I := K_{i_1} \times \ldots \times K_{i_n}$ by

$$(k_1, \ldots, k_n) \cdot (a_1, \ldots, a_n) = (k_1 a_1, a_1^{-1} k_2 a_2, \ldots, a_{n-1}^{-1} k_n a_n). \tag{3.8}$$

The inclusion map

$$K_I = K_{i_1} \times \ldots \times K_{i_n} \hookrightarrow \mathbf{P}_I = P_{i_1} \times \ldots \times P_{i_n}$$

is equivariant with respect to the actions (3.8) and (3.6) and the inclusion $T^{(n)} \hookrightarrow B^n$, because $\Upsilon_t(a_i) = a_i$ for all $a_i \in T$. Therefore we get a map $K_I/T^{(n)} \longrightarrow M_I^t$. This map is a diffeomorphism; this follows from the fact that for all i, the inclusion $K_i \hookrightarrow P_i$ induces a diffeomorphism $K_i/T \cong P_i/B$.

Similarly, the bundles $\mathbf{L}^t_{I,\lambda_1,\ldots,\lambda_n}$ are equivalent to the bundle $K_I \times_{T^{(n)}} \mathbb{C}_{\lambda_1,\ldots,\lambda_n}$ where $T^{(n)}$ acts on $\mathbb{C}_{\lambda_1,\ldots,\lambda_n}$ as in §3.2. $\square$

Thus t represents a parameter of complex structures on the smooth manifold M_I. Moreover for $t < \infty$ these structures are equivalent, because the map

$$(p_1, \ldots, p_n) \mapsto (p_1, \exp(-t\underline{h})p_2, \ldots, \exp(-t\underline{h})p_n)$$

descends to a biholomorphism $M_I^0 \longrightarrow M_I^t$.

Remark 3.9. Our family of complex structures is part of a larger family, indexed by *all* $\underline{h} \in \mathfrak{h}$, which is constructed in the same way from the homomorphisms $\Upsilon_{\exp(\underline{h})}$. In a subsequent paper, the first author exploits these families of deformations to obtain systems of filtrations of the space of holomorphic sections over the Bott-Samelson varieties with the ordinary complex structure. This gives more refined information than the index. In fact, results have been obtained which are formally similar to the Kashiwara/Lustig construction of the crystal base/canonical basis, see [G-Z].

3.5. **Connections with Bott Towers.** Let $I = \{i_1, \ldots, i_n\}$ and let $I' = \{i_1, \ldots, i_{n-1}\}$; then there is a natural map, $\mathbf{P}_I \longrightarrow \mathbf{P}_{I'}$, given by projection onto the first $n-1$ factors. This map is equivariant with respect to the R_t actions (3.6) on $\mathbf{P}_I$ and on $\mathbf{P}_{I'}$, therefore it descends to a map $\pi : M_I^t \longrightarrow M_{I'}^t$ for all t. The fiber of this map is $P_{i_n}/B \cong \mathbb{C}P^1$, thus M_I^t is a $\mathbb{C}P^1$-bundle over $M_{I'}^t$. By induction, M_I^t is a successive fibration of $\mathbb{C}P^1$ bundles, i.e., it is a $\mathbb{C}P^1$ over a $\mathbb{C}P^1$, etc. This is very close to M_I^t being a Bott tower; in fact,

Proposition 3.10. M_I^∞ *is a Bott tower.*

Proof. If $n = 1$ then $M_I^\infty = P_i/B$ where $\alpha = \alpha_i$ is a simple root. This is isomorphic to $\mathbb{C}P^1$ in the following way.

Let $\underline{h}_\alpha \in \mathfrak{h}$ be the co-root which corresponds to α, via the Killing form, and let $\alpha^\vee = 2\underline{h}_\alpha/\langle\alpha,\alpha\rangle$. We can choose $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = \alpha^\vee$. We get a Lie algebra homomorphism $sl(2,\mathbb{C}) \longrightarrow \mathfrak{p}_\alpha$ by sending $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mapsto X_\alpha$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \mapsto X_{-\alpha}$, and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mapsto \alpha^\vee$. This descends to a map $\psi : SL(2,\mathbb{C}) \longrightarrow P_\alpha$ and further to a holomorphic diffeomorphism

$$SL(2,\mathbb{C})/B_{SL(2,\mathbb{C})} \xrightarrow{\cong} P_\alpha/B \tag{3.11}$$

where $B_{SL(2,\mathbb{C})}$ denotes the lower triangular matrices. Moreover, the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ $\mapsto [d,b]$ descends to a holomorphic diffeomorphism

$$SL(2,\mathbb{C})/B_{SL(2,\mathbb{C})} \xrightarrow{\cong} \mathbb{C}P^1 \tag{3.12}$$

in which the identity coset is sent to the South pole $[1,0]$ in $\mathbb{C}P^1$. The composition of (3.12) with the inverse of (3.11) gives the desired diffeomorphism $P_\alpha/B \cong \mathbb{C}P^1$.

We proceed by induction. Let $I = \{i_1, \ldots, i_n\}$ and let $I' = \{i_1, \ldots, i_{n-1}\}$. We need to find a holomorphic line bundle $\mathbf{L}_n$ over $M_{I'}^\infty$ such that $M_I^\infty = \mathbb{P}(\mathbf{1} \oplus \mathbf{L_n})$. Recall that $M_{I'}^\infty = \mathbf{P}_{I'}/B^{n-1}$ with respect to the action R_∞, and

$$M_I^\infty = (P_{i_1} \times_{B_\infty} \ldots \times_{B_\infty} P_{i_{n-1}}) \times_{B_\infty} (P_{i_n}/B).$$

Let $\mathbf{L}_n = \mathbf{P}_{I'} \times_{B^{n-1}} \mathbb{C}_{\alpha_{i_n}}$ where only the last B factor acts on the B-module $\mathbb{C}_{\alpha_{i_n}}$. Then

$$\mathbb{P}(\mathbf{1} \oplus \mathbf{L_n}) = (\mathbf{P_{i_1}} \times_{\mathbf{B_\infty}} \ldots \times_{\mathbf{B_\infty}} \mathbf{P_{i_{n-1}}}) \times_{\mathbf{B_\infty}} \mathbb{P}(\mathbb{C} \oplus \mathbb{C}_{\alpha_{\mathbf{i_n}}}).$$

Therefore, it is sufficient to find a left B-equivariant map $\hat{\psi} : P_{i_n}/B \longrightarrow \mathbb{P}(\mathbb{C} \oplus \mathbb{C}_{\alpha_{i_n}})$.

We take the map $\hat{\psi}$ which was constructed for the case $n = 1$, with $\alpha = \alpha_{i_n}$. The unipotent part of B acts trivially on both P_{i_n}/B and $\mathbb{P}(\mathbb{C} \oplus \mathbb{C}_{\alpha_{i_n}})$ by the definitions of those actions. Therefore it is enough check equivariance with respect to the actions of H.

The map (3.12) is equivariant with respect to the following actions of the Cartan $H_{SL(2,\mathbb{C})}$. It acts on $SL(2,\mathbb{C})/B_{SL(2,\mathbb{C})}$ by left multiplication. Its action on $\mathbb{C}P^1$ is induced from a linear action on $\mathbb{C}^2$ with weights $(-\gamma, \gamma)$ where $\gamma(Z) = 1$ for $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Then, the map (3.11) is equivariant with respect to the left actions of $H_{SL(2,\mathbb{C})}$ and of H, where we map $H_{SL(2,\mathbb{C})} \longrightarrow H$ by sending Z to $\alpha^\vee$. Dually, the root α of P_α pulls-back to the root 2γ of $SL(2,\mathbb{C})$.

Combining these actions, we get that H acts on $\mathbb{C} \oplus \mathbb{C}$ with weights $(-\frac{1}{2}\alpha, \frac{1}{2}\alpha)$, and $\mathbb{C}P^1 = \mathbb{P}(\mathbb{C}_{-\frac{1}{2}\alpha} \oplus \mathbb{C}_{\frac{1}{2}\alpha}) = \mathbb{P}(\mathbb{C} \oplus \mathbb{C}_\alpha)$ as we wanted. $\square$

Note that the South pole section is then $\sigma_n^S : [p_1, \ldots, p_{n-1}] \mapsto [p_1, \ldots, p_{n-1}, e]$.

3.6. **The torus actions.** Recall that the maximal compact torus $T \subset G$ acts on $M_I = P_{i_1} \times_B \cdots \times_B P_{i_n}/B$ by the inclusion $T \hookrightarrow P_{\alpha_{i_1}}$ followed by left multiplication. This action is holomorphic on M_I^t for all t. The multiplication map (3.3) $\eta : M_I \longrightarrow X_w$ is T-equivariant and is holomorphic with respect to the complex structure at $t = 0$.

We define an action of $T^{(n)} = T \times T \times \ldots \times T$ on M_I^∞ by

$$\begin{aligned} \mathbf{a} \cdot [p] &= [a_1 p_1, a_1^{-1} a_2 p_2, \ldots, a_{n-1}^{-1} a_n p_n] \\ &= [a_1 p_1 a_1^{-1}, \ldots, a_n p_n a_n^{-1}]. \end{aligned}$$

This is well defined and holomorphic on M_I^∞. If we embed $T \hookrightarrow T^{(n)}$ by the diagonal map then this induces the same T-action on M_I^∞ as before.

Let $T_\alpha = \exp(\ker \alpha)$, then the subgroup $T_I := T_{\alpha_{i_1}} \times \ldots \times T_{\alpha_{i_n}} \hookrightarrow T^{(n)}$ acts trivially on M_I^∞ and we get an action of the quotient $T^{(n)}/T_I$. We will show that this is a complete torus action as defined in §2.4. Thus at $t = \infty$, the left T-action extends to a complete torus action on a Bott tower. Note that the action of

$T^{(n)}/T_I$ is defined on every M_I^t by Proposition 3.7, and at $t = \infty$ we know that it is holomorphic.

Now, consider the line bundle $\mathbf{L}_{I,\lambda} = P_{i_1} \times_B \cdots \times_B P_{i_n} \times_B \mathbb{C}_\lambda$ over M_I as in §3.2. An action of $T^{(n+1)} = T \times \ldots \times T$ ($n+1$ factors) on $\mathbf{L}_{I,\lambda}$ is given by

$$\mathbf{a} \cdot [\mathbf{p}, v] = [a_1 p_1, a_1^{-1} a_2 p_2, \ldots, a_{n-1}^{-1} a_n p_n, a_n^{-1} a_{n+1} v] \tag{3.13}$$

for $\mathbf{a} \in T^{(n+1)}$ and $[\mathbf{p}, v] \in \mathbf{L}_{I,\lambda}$. Again this is defined and is holomorphic for $t = \infty$, and is defined for $t < \infty$ via Proposition 3.7. Again, the map $\tilde{\eta}$ in (3.2) is T-equivariant when we embed $T \hookrightarrow T^{(n+1)}$ diagonally. Again, the subgroup $T_{I,\lambda} = T_{\alpha_{i_1}} \times \ldots \times T_{\alpha_{i_n}} \times T_\lambda$ acts trivially. We will assume that $\lambda \neq 0$; the other case can be treated with some careful bookkeeping but the details are not illuminating. Consider the quotient $\tilde{T} = T^{(n+1)}/T_{I,\lambda}$.

Proposition 3.14. *The action of $\tilde{T}$ on $\mathbf{L}_{I,\lambda}^\infty \longrightarrow M_I^\infty$ is a complete torus action on a Bott tower as defined in §2.4.*

Proof. First, the action (3.13) is holomorphic on $\mathbf{P}_I$ and commutes with the action R_∞, thus it descends to a holomorphic action on $\mathbf{L}_{I,\lambda}^\infty$. Next, it preserves the north and south pole maps because at each stage of the tower, the fiber can be identified with $K_{i_j}/T \cong \mathbb{C}P^1$ as in §3.5 and the left T-action preserves its north and south poles. Next, $\tilde{T}$ is an $(n+1)$-torus because it can be written as $T/T_{\alpha_{i_1}} \times \ldots \times T/T_{\alpha_{i_n}} \times T/T_\lambda$. Finally, $\tilde{T}$ acts infinitesimally effectively as one can check by induction on n. $\square$

3.7. **Computation of the Bott-tower integers.** We will now compute the integers $\{c_{ij}\}$ and $\{l_i\}$ corresponding to the Bott tower M_I^∞, as well as the map $\tilde{T} \longrightarrow T^{n+1}$ which gives the torus action as in §2.4. We start with $n = 1$; suppose $I = \{i\}$ and $\alpha = \alpha_i$. We want to compute the Chern number l_1 of the line bundle $L_{I,\lambda} = P_i \times_B \mathbb{C}_\lambda$. We reduce to the case of $SL(2,\mathbb{C})$ by considering the map $\psi : SL(2,\mathbb{C}) \longrightarrow P_\alpha$ which was constructed in §3.5. Remember, the roots of $SL(2,\mathbb{C})$ were $\pm 2\gamma$ with $\langle \gamma, Z \rangle = 1$ for $Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$. We have that $\langle \psi^* \lambda, Z \rangle = \langle \lambda, \psi_* Z \rangle = \langle \lambda, \alpha^\vee \rangle = \langle \langle \lambda, \alpha^\vee \rangle \gamma, Z \rangle$, thus $\psi^* \lambda = \langle \lambda, \alpha^\vee \rangle \gamma$, so the weight e^λ on H pulls back to the weight $e^{\langle \lambda, \alpha^\vee \rangle \gamma}$ on $H_{SL(2,\mathbb{C})}$. Thus $\mathbf{L}_\lambda \cong SL(2,\mathbb{C}) \times_{B_\infty} \mathbb{C}_{\langle \lambda, \alpha^\vee \rangle \gamma}$, i.e., in $\mathbf{L}_\lambda$ we have $\left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \left(\begin{smallmatrix} x & 0 \\ y & x^{-1} \end{smallmatrix} \right), v \right] = \left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), x^{\langle \lambda, \alpha^\vee \rangle} v \right]$. This is equivalent to the bundle over $\mathbb{C}P^1$ given by $[dx^{-1}, bx^{-1}, v] = [d, b, x^{\langle \lambda, \alpha^\vee \rangle} v]$ as in §2.2. Its Chern number is $l_1 = \langle \lambda, \alpha^\vee \rangle$.

A similar computation works for the line bundle $\mathbf{L}_{I,\lambda_1,\ldots,\lambda_n} = \mathbf{P}_I \times_{B_\infty^n} \mathbb{C}_{\lambda_1,\ldots,\lambda_n}$ over M_I^∞. We form the map

$$\begin{array}{rcl} \psi_j : SL(2,\mathbb{C}) & \longrightarrow & P_{i_1} \times \cdots \times P_{i_j} \times \cdots \times P_{i_n} \\ p & \mapsto & (e, \ldots, e, \psi(p), e, \ldots, e) \end{array}$$

where ψ is the same as before, with $\alpha = \alpha_{i_j}$. The map ψ_j is equivariant via the map

$$\begin{array}{rcl} B_{SL(2,\mathbb{C})} & \longrightarrow & B \times \ldots \times B \\ b & \mapsto & (e, \ldots, e, \psi(b), \psi(h), \ldots, \psi(h)) \end{array} \tag{3.15}$$

when $h = \Upsilon_\infty(b) \in H_{SL(2,\mathbb{C})}$ as in §3.4, where $B_{SL(2,\mathbb{C})}$ acts on $SL(2,\mathbb{C})$ by right multiplication, and where $B \times \ldots \times B$ acts on P_I by the action R_∞ (3.6). Thus ψ_j

descends to $\hat{\psi}_j : SL(2,\mathbb{C})/B_{SL(2,\mathbb{C})} \longrightarrow M_I^\infty$ whose image is the sphere $S_j^{(n)}$ defined in §2.3.

By the map (3.15), the weight $(\lambda_1, \dots, \lambda_n)$ pulls back to $\langle \lambda_j + \dots + \lambda_n, \alpha_{i_j}^\vee \rangle \gamma$. Thus $\mathbf{L}_{I,\lambda_1,\dots,\lambda_n}$ pulls back to a line bundle with Chern class $l_j = \langle \lambda_j + \dots + \lambda_n, \alpha_{i_j}^\vee \rangle$.

As for the integers $c_{j\ell}$, recall from §3.5 that the ℓth step of the tower was constructed from the line bundle $\mathbf{L}_{I,0,\dots,0,\alpha_{i_\ell}}$ over the $(\ell-1)$-step tower. Thus by the above computation for line bundles, $c_{j\ell} = \langle \alpha_{i_\ell}, \alpha_{i_j}^\vee \rangle$.

The complete action of

$$\tilde{T} = T/T_{\alpha_{i_1}} \times \dots \times T/T_{\alpha_{i_n}} \times T/T_\lambda \tag{3.16}$$

on $\mathbf{L}_{I,\lambda}$ can be described by a map $A : \tilde{T} \longrightarrow T^{n+1} = S^1 \times \dots \times S^1$ followed by the standard action of T^{n+1}; see §2.4. Let $f_1, \dots, f_{n+1}$ be the standard basis in $i\mathrm{Lie}\,(T^{n+1})^* \cong \mathbb{R}^{n+1}$. Then for $1 \le j \le n$, $\beta_j = A^* f_j$ is the weight by which $\tilde{T}$ rotates the sphere $S_j^{(n)}$ (in the notation of §2.3); and $\beta_{n+1} = A^* f_{n+1}$ is the weight by which $\tilde{T}$ acts on the fiber of $\mathbf{L}_{I,\lambda}$ over the point $[S, \dots, S]$ (in the notation of §2.3).

Equation (3.16) lets us identify $i\tilde{\mathfrak{t}}^*$ with $\mathbb{R}^{n+1}$. With this identification we have $\beta_j = (0, \dots, 0, 2, 0, \dots, 0)$ for $1 \le j \le n$ (with the 2 in the jth place) and $\beta_{n+1} = (0, \dots, 0, m)$ where m is the maximal integer for which λ/m is an integral weight.

Also note that via $T \xrightarrow{diag} \tilde{T} \xrightarrow{A} T^{n+1}$, the pull-back of f_j is α_{i_j} for $1 \le j \le n$ and the pull-back of f_{n+1} is λ.

3.8. **Application to the symplectic picture.** Let X be a flag manifold, realized as a coadjoint orbit through $\lambda' \in \mathfrak{t}^*$ as in §1.3. Let ω be the symplectic form on X and let $\Phi : X \longrightarrow \mathfrak{t}^*$ be a corresponding moment map. Motivated by §1.6, we wish to describe the D-H measure $\Phi_* \omega^n$, where $n = \dim_{\mathbb{C}} X$. This measure is supported on *image* (Φ), which is a convex polytope, in which the regular values of Φ form polyhedral regions [A, G-S 1]. On each of these regions, the measure $\Phi_*\omega^n$ is given by a polynomial density function times Lebesgue measure [D-H].

By using the machinery developed in previous sections, we will now give a description of $\Phi_*\omega^n$ which will illuminate the facts mentioned above. Moreover, this description will also apply in the case that we replace X by a Schubert variety X_w; even if it is singular, when the Duistermaat Heckman theorem does not apply. The D-H measure is then defined by working not with all of X_w but only with the Bruhat cell whose closure is X_w. We then denote by ω' the symplectic form on the coadjoint orbit X, we take ω to be the pullback of ω' to the Bruhat cell under the inclusion map, n is the dimension of the Bruhat cell, and the moment map Φ is the inclusion of the Bruhat cell into $\mathfrak{k}^*$, composed with the projection $\mathfrak{k}^* \longrightarrow \mathfrak{t}^*$ as in §1.4.

Theorem 4. *There is a twisted cube C in $\mathbb{R}^n$ and an affine projection $L : \mathbb{R}^n \longrightarrow \mathfrak{t}^*$ such that $\Phi_*\omega^n = L_* m_C$ in the notation of §2.5.*

Proof. Consider the maps $M_I \xrightarrow{\eta} X_w \xrightarrow{\Phi} \mathfrak{t}^*$ where M_I is the Bott-Samelson manifold associated to a Schubert variety X_w and to a reduced expression I of w as in §3.3. Consider the 2-form $\eta^*\omega$. If X_w is singular then ω is only defined on a dense open set, but the pullback $\eta^*\omega$ extends smoothly to all of M_I, because it is equal to the pullback of ω' under the multiplication map $M_I \xrightarrow{\eta} X$ whose image was X_w. The composition $\Phi \circ \eta$ is a moment map for $(M_I, \eta^*\omega, T)$; this follows easily from the definition of the moment map. Note that the pull-back $\eta^*\omega$ is no longer symplectic.

Consider the action on M_I of the big torus, $\tilde{T} = T^{(n)}/T_I \cong (S^1)^n$, given in §3.6. We have an inclusion $T \hookrightarrow \tilde{T}$ and, dually, $\tilde{\mathfrak{t}}^* \xrightarrow{L} \mathfrak{t}^*$. Note that $\tilde{\mathfrak{t}}^* = \mathbb{R}^n$ naturally.

For any closed and invariant 2-form $\tilde{\omega}$ on $(M_I, \tilde{T})$ and a corresponding moment map $\tilde{\Phi} : M_I \longrightarrow \mathbb{R}^n$, the push-forward measure $\tilde{\Phi}_*\tilde{\omega}^n$ is a twisted cube in $\mathbb{R}^n$; this follows from §2.6 and Propositions 3.10 and 3.14. We would then like to apply Remark 1.4. Unfortunately, $\eta^*\omega$ might not be $\tilde{T}$-invariant. We therefore take its average,

$$\tilde{\omega} = \int_{a \in \tilde{T}} (a^*(\eta^*\omega))\, da.$$

Then $\tilde{\omega}$ is a pre-symplectic form on $(M_I, \tilde{T})$. Also, $\tilde{\omega}$ and $\eta^*\omega$ represent the same cohomology class in $H^2(M_I)$. Let $\tilde{\Phi}$ be a moment map for $(M_I, \tilde{T}, \tilde{\omega})$. Then $\tilde{\Phi}_*\tilde{\omega}^n = m_C$ where C is a twisted cube in $\tilde{\mathfrak{t}}^*$. The diagram below

$$\begin{array}{ccc} M_I & \xrightarrow{\tilde{\Phi}} & \mathbb{R}^n \\ \eta \downarrow & \begin{array}{c}\text{does not}\\ \text{commute}\end{array} & \downarrow L \\ X_w & \xrightarrow{\Phi} & \mathfrak{t}^* \end{array}$$

does not commute, but $L \circ \tilde{\Phi}$ and $\Phi \circ \eta$ are moment maps for $(M_I, T, \tilde{\omega})$ and for $(M_I, T, \eta^*\omega)$ respectively. By Theorem 1, the push-forward of Liouville measure only depends on the cohomology class $[\tilde{\omega}] = [\eta^*\omega]$, so

$$(L \circ \tilde{\Phi})_*\tilde{\omega}^n = (\Phi \circ \eta)_*(\eta^*\omega)^n. \tag{3.17}$$

Thus we have $\Phi_*\omega^n/n! = L_* m_C$. □

Corollary 3.18. Theorem 4 exhibits the measure $\Phi_*\omega^n/n!$ as the linear projection of a twisted cube C. This sheds light on the polynomial nature of $\Phi_*\omega^n$ which was established by Duistermaat and Heckman. Indeed, the density function for $\Phi_*\omega^n/n!$ is given by $\rho(\alpha) = vol(p^{-1}(\alpha) \cap C)$ where we take a "twisted volume", i.e., we integrate the density function of C over the set $p^{-1}(\alpha)$. The set $C_\alpha := p^{-1}(\alpha) \cap C$ is a finite union of polytopes, each with a density function equal to 1 or -1, bounded by some hyperplanes. As α varies in a region of regular values of Φ, the components of C_α change in that their bounding hyperplanes get parallel-translated. Moreover, the location of the hyperplanes depends linearly on α. It follows that $\rho(\alpha) = vol(C_\alpha)$ depends on the location of the faces in a polynomial manner. One way to see this is by investigating the dependence of $vol(C_\alpha)$ on the location of each face F. Suppose that F lies on the hyperplane $\varphi_F = c_F$ for some linear functional φ_F, then one can show that $\frac{\partial}{\partial c_F} vol(C_\alpha) = vol(F)$. By induction, $vol(F)$ is a polynomial of degree $\dim \tilde{T} - \dim T - 1$ in the variables $\{c_{F'} \mid \textit{the face } F' \textit{ intersects } F\}$. Thus $vol(C_\alpha)$ is a polynomial of degree $\dim \tilde{T} - \dim T$ in c_F and the $c_{F'}$'s.

3.9. **Application to the index.** In §3.8 we related the symplectic form on a flag manifold to an invariant 2-form on a Bott-tower by pulling-back and averaging. We would like to obtain a similar relation between the holomorphic structures. The analogue of pulling-back is that we pass from a flag manifold to a Bott-Samelson manifold, which we can do by Demazure's theorem (3.4). The averaging should be replaced by some process which allows us to pass from the index over the Bott-Samelson manifold to the index over a Bott tower. This we will get from the invariance of the index under deformations. We now carry out this program.

Let M be a complex manifold. Then we have the Dolbeault complex;

$$\Omega^{0,0}(M) \xrightarrow{\bar\partial} \Omega^{0,1}(M) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{0,n}(M).$$

Let $\mathbf{L} \longrightarrow M$ be a holomorphic line bundle. The choice of a Hermitian metric on $\mathbf{L}$ determines a connection

$$\nabla = \nabla' + \nabla'' : \Gamma(\mathbf{L}) \longrightarrow \Omega^{1,0}(M,\mathbf{L}) \oplus \Omega^{0,1}(M,\mathbf{L})$$

where $\Omega^{i,j}(M,\mathbf{L}) = \Omega^{i,j}(M) \otimes \Gamma(\mathbf{L})$. From this we get a complex

$$\Omega^{0,0}(M,\mathbf{L}) \xrightarrow{\overline{\partial}_{\mathbf{L}}} \Omega^{0,1}(M,\mathbf{L}) \xrightarrow{\overline{\partial}_{\mathbf{L}}} \cdots \xrightarrow{\overline{\partial}_{\mathbf{L}}} \Omega^{0,n}(M,\mathbf{L})$$

where $\overline{\partial}_{\mathbf{L}} = \overline{\partial} \otimes 1 \pm 1 \otimes \nabla''$. The cohomology of this complex is equal to the sheaf cohomology $H^*(M, \mathcal{O}(\mathbf{L}))$. We denote either of these cohomologies by $H^*(M,\mathbf{L})$.

If we choose a Hermitian metric on M then we can form the operator $\mathcal{D} = (\bar\partial + \bar\partial^*) \otimes 1 + 1 \otimes \nabla''$. This is an elliptic differential operator which sends $\Omega^{0,even}(M,\mathbf{L})$ to $\Omega^{0,odd}(M,\mathbf{L})$ and whose kernel and cokernel are equal to

$$\begin{array}{rcl} \mathrm{Ker}(\mathcal{D}) & = & \bigoplus_j H^{2j}(M,\mathbf{L}) \\ \mathrm{CoKer}(\mathcal{D}) & = & \bigoplus_j H^{2j+1}(M,\mathbf{L}) \end{array} \tag{3.19}$$

Suppose that a torus T acts holomorphically on $\mathbf{L} \longrightarrow M$ and preserves the metrics, then (3.19) holds as equalities between T-representations. We can then form the equivariant index, $index_T(\mathcal{D}) = \ker(\mathcal{D}) - \mathrm{coker}(\mathcal{D}) = \sum(-1)^j H^j(M,\mathbf{L})$, as in §2.7. Recall, this is a virtual T-representation, determined by its virtual character $\chi : T \longrightarrow \mathbb{C}$, or equivalently, by its multiplicity function $mult : \ell^* \longrightarrow \mathbb{Z}$; see §2.7.

The advantage of working with the index of an elliptic operator rather than sheaf cohomology lies in the following theorem.

Theorem. *(See* [L-M, p.214, cor.9.6]*).* The equivariant index of $\mathcal{D}$ only depends on the homotopy class of $\mathcal{D}$ in the space of T-equivariant elliptic operators from $\Omega^{0,\mathrm{even}}(M,\mathbf{L})$ to $\Omega^{0,\mathrm{odd}}(M,\mathbf{L})$.

We now specialize to the case that $M = M_I^t$ and $\mathbf{L} = \mathbf{L}_{I,\lambda}^t$ as constructed in §3.4. Then we get a whole family of operators $\mathcal{D}_t$ for $t \le \infty$. The bundles $(\bigwedge^{0,\mathrm{even/odd}} T^*M_I \otimes \mathbf{L}_I)_t$ are all smoothly isomorphic so we can redefine $\mathcal{D}_t$ so that they have the same domain and range. Then, by the above theorem, the index of $\mathcal{D}_t$ is independent of t. Denote its character by $\boldsymbol{\chi}_t : T \longrightarrow \mathbb{C}$.

Recall, when $t = \infty$, the action of T extends to a holomorphic action of a larger torus $\tilde T$ on $\mathbf{L}_{I,\lambda}^\infty \longrightarrow M_I^\infty$. Denote the corresponding virtual character by $\tilde{\boldsymbol{\chi}} : \tilde T \longrightarrow \mathbb{C}$, then its restriction to T is $\boldsymbol{\chi}_\infty$. Recall, M_I^∞ is a Bott tower and $\tilde T$ acts by a complete torus action as defined in §2.4. Thus the descriptions of the index in §2.7 apply to $\tilde{\boldsymbol{\chi}}$.

Proposition 3.20. *Let $w \in W$ be the longest element of the Weyl group. Let $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_n}}$ be a minimal expression for w in terms of simple reflections and let $I = (i_1, \ldots, i_n)$. Let λ be a dominant integral weight. Let $\tilde{\boldsymbol{\chi}} : \tilde T \longrightarrow \mathbb{C}$ be the virtual character for the action of $\tilde T$ on M_I^∞ and $\mathbf{L}_{I,\lambda}^\infty$. Recall that we have an embedding $T \hookrightarrow \tilde T$. Then the restriction of $\tilde{\boldsymbol{\chi}}$ to T coincides with the character $\boldsymbol{\chi}_T(\lambda)$ of the irreducible representation R_λ of G of highest weight λ.*

Proof. From the above discussion we have, at $t = \infty$, that $\tilde{\mathcal{X}}|_T = \mathcal{X}_\infty$. Demazure's theorem (3.4) together with the Borel-Weil theorem tell us that $H^i(M_I, \mathbf{L}_{I,\lambda})$ vanishes for $i > 0$ and that $H^0(M_I, \mathbf{L}_{I,\lambda})$ is the irreducible representation of G with highest weight λ, thus at $t = 0$ we have $\mathcal{X}_0 = \mathcal{X}_T(\lambda)$. Finally, by the homotopy invariance of the index, $\mathcal{X}_0 = \mathcal{X}_\infty$. $\square$

Theorem 5. *Let $C \subseteq \mathbb{R}^n$ be the twisted cube corresponding to the Bott-tower M_I^∞ and the line bundle $L_{I,\lambda}^\infty$. Denote by $L : \mathbb{R}^n \longrightarrow \mathfrak{t}^*$ the linear projection which is dual to the inclusion $T \hookrightarrow \tilde{T}$. Then the multiplicity of a weight $\alpha \in \ell^*$ in the representation R_λ is equal to the number of lattice points, counted with signs, in the polyhedral region $C \cap L^{-1}(i^{-1}\alpha)$.*

Proof. This follows immediately from Proposition 3.20, Theorem 3, Proposition 3.10 and Remark 2.28. $\square$

We call $\tilde{\mathcal{X}}$ the *extended character.* We should point out that we have made two unnecessary specializations for clarity. First, we stated the results for w the longest element of W because this corresponds to representations of G. However, one can also state the results for an arbitrary element of W. In this case one would obtain extended characters of certain representations of B associated to Schubert Varieties. Second, we assumed that λ is dominant. Thanks to the Borel-Weyl-Bott theorem, the case of non-dominant λ can easily be treated.

Remark 3.21. We can interpret Theorem 5 pictorially as follows. If we choose a basis for $\tilde{T}$ then a weight of $\tilde{T}$ is represented by n integers. For example if $K = SU(3)$, and $w_0 = w$ is the longest element then in terms of the simple reflections s_{α_1} and s_{α_2} there is a reduced expression $w = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$. Thus $n = 3$ and so $i\tilde{\mathfrak{t}}$ is three dimensional. We now plot the *twisted cube* for $\lambda = 3\gamma_1 + 4\gamma_2$ where the γ_i are fundamental weights.

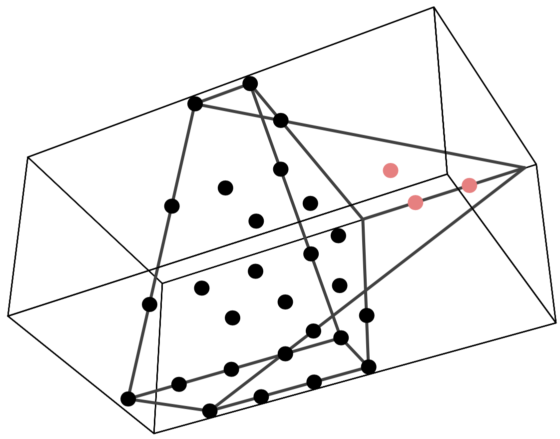

FIGURE 2. $K = SU(3)$, Twisted cube

The darker dots represent $H^{\text{even}}(M_I^\infty, \mathbf{L}_{I,\lambda})$ and the lighter dots represent $H^{\text{odd}}(M_I^\infty, \mathbf{L}_{I,\lambda})$. In this case, for λ dominant, we have $H^2(M_I^\infty, \mathbf{L}_{I,\lambda}) = H^3(M_I^\infty, \mathbf{L}_{I,\lambda}) = 0$, so 'even' is really just the 0th cohomology, and 'odd' is just the first cohomology.

The lines represent the twisted cube defined by equations (2.21). We take the lattice points which satisfy equations (2.21).

Proposition 3.20 is interpreted by looking at the projection $i\tilde{\mathfrak{t}}^* \longrightarrow i\mathfrak{t}^*$ dual to the inclusion. If we project figure 2 using L, with L as in Theorem 4 we get

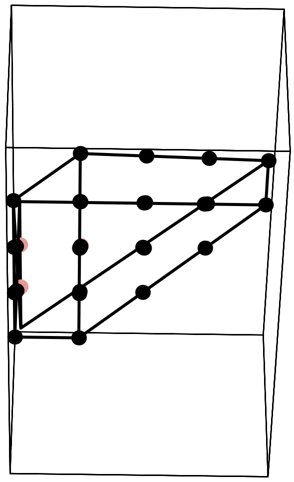

FIGURE 3. Twisted cube projected into $\mathfrak{t}^*$

which is the familiar weight diagram for a generic representation of $SU(3)$. We see that for a weight $\mu \in i\mathfrak{t}^*$ the multiplicity is the number of preimage points in the twisted cube, counted with sign. Again this bares strong similarities with the work of Kashiwara and Lusztig in that, by choosing a decomposition of the longest element, they can parameterizes their crystal basis (resp. canonical basis) by a lattice points in a convex body defined by equations similar to equations (2.21), e.g., [Lu, §8.12].

We note that the multiplicities can also be given by the famous formula of Kostant [Ko2, (1.1.5)]. From that formula alone it might be suprising that the region for which the multiplicity is non-zero is bounded. This fact is obvious from the definition of the character and the finite dimensionality of the representation, and it can be seen geometrically in the description that we gave above. We note that in Kostant's paper, he defines a function "Q" which bares some kinship to our formula, in that it is defined by a sum of ± 1 over a finite combinatorially defined region.

3.10. **A Demazure type formula.** As a corollary of Proposition 3.20 and Theorem 3 we will now prove a Demazure type character formula. This formula bares strong formal resemblances to the formula of Littelmann regarding the crystal base of Kashiwara, see Remark 3.23. In fact, our formula implies Demazure's formula.

First let us recall the Demazure formula [D], [J, II,§14.17]. Suppose that $\lambda \in i\mathfrak{t}^*$ is a dominant weight. Let $\chi_T(\lambda)$ be the character of the representation of highest weight λ restricted to the maximal torus T as before. Denote by $\mathbb{Z}[T]$ the space of functions on T which are obtained as integral combinations of the multiplicative weights, e^μ, for $\mu \in \ell^*$. For each simple root α define the operators $\mathbf{D}_\alpha : \mathbb{Z}[T] \longrightarrow \mathbb{Z}[T]$ as follows.

$$\mathbf{D}_\alpha(e^\mu) = \begin{cases} e^\mu + e^{\mu-\alpha} + \ldots + e^{\mu - \langle \mu, \alpha^\vee \rangle \alpha} & \text{if } \langle \mu, \alpha^\vee \rangle \geq 0 \\ 0 & \text{if } \langle \mu, \alpha^\vee \rangle = -1 \\ -e^{\mu+\alpha} - \ldots - e^{\mu - (1 + \langle \mu, \alpha^\vee \rangle)\alpha} & \text{if } \langle \mu, \alpha^\vee \rangle \leq -2 \end{cases}$$

$\mathbf{D}_\alpha$ extends to $\mathbb{Z}[T]$ by additivity. Demazure's formula says that if $\alpha_{i_1}, \ldots, \alpha_{i_n}$ is a sequence of simple roots associated to a reduced expression of the longest element of the Weyl group, then

$$\chi_T(\lambda) = \mathbf{D}_{\alpha_{i_1}} \cdots \mathbf{D}_{\alpha_{i_n}} e^{\lambda}.$$

Note that the coefficient of e^{μ} in this expression is also given by Kostant's multiplicity formula [Ko2].

We can now state our theorem. In what follows, The β_j's are as in the notation of Proposition 2.33, when we consider the homomorphism $\tilde{T} \longrightarrow T^{n+1}$ which gives the action of $\tilde{T}$ on the Bott-tower. In §3.7 we computed the β_j's explicitly. Note that they form a linear basis for $i\tilde{\mathfrak{t}}^*$. We also computed the pullbacks of the β_j's to $i\mathfrak{t}^*$ via the inclusion $T \hookrightarrow \tilde{T}$. Denote by $L : \tilde{\mathfrak{t}}^* \longrightarrow \mathfrak{t}^*$ the dual projection, then we obtained $L(\beta_j) = \alpha_{i_j}$ for $1 \le j \le n$ and $L(\beta_{n+1}) = \lambda$.

Theorem 6. *Let $\tilde{\chi}^i$ the character of $\tilde{T}$ acting on $H^i(M_I^{\infty}, \mathbf{L}_{I,\lambda})$ and let $\tilde{\chi} = \sum (-1)^i \tilde{\chi}^i$. As an element of $\mathbb{Z}[\tilde{T}]$, it is given by*

$$\tilde{\chi} = \hat{D}_1 \cdots \hat{D}_n e^{\beta_{n+1}}$$

where we define $\hat{D}_j : \mathbb{Z}[\tilde{T}] \longrightarrow \mathbb{Z}[\tilde{T}]$ by

$$\hat{D}_j e^{\mu} = \begin{cases} e^{\mu} + e^{\mu-\beta_j} + \cdots + e^{\mu - k_j \beta_j} & \text{if} \quad k_j \ge 0 \\ 0 & \text{if} \quad k_j = -1 \\ -e^{\mu+\beta_j} - \cdots - e^{\mu-(1+k_j)\beta_j} & \text{if} \quad k_i \le -2 \end{cases}$$

and where $k_i = k_i(\mu)$ are defined for as follows; if $\mu = \sum x_j \beta_j$ then $k_i(\mu) = \langle \lambda, \alpha_{i_j}^{\vee} \rangle + \sum_{l=j+1}^{n} \langle \alpha_{i_l}, \alpha_{i_j}^{\vee} \rangle x_l$.

Proof. The formula follows from Proposition 2.33, that for the bundle $\mathbf{L}_{I,\lambda}^{\infty} \longrightarrow M_{\lambda}^{\infty}$ we have $l_j = \langle \lambda, \alpha_{i_j}^{\vee} \rangle$ and $c_{jl} = \langle \alpha_{i_l}, \alpha_{i_j}^{\vee} \rangle$; see §3.7. □

Remark 3.22. We can simplify the definitions of the k_j's in Theorem 6 in the following way. Note that $k_i(\mu) = \langle \sum_{l=1}^{n} \alpha_{i_l} x_l + \lambda x_{n+1}, \alpha_{i_j}^{\vee} \rangle$, this follows from the formula for the extended character $\tilde{\chi}$, because we only apply k_j to μ's which are combinations of $\beta_{j+1}, \dots, \beta_{n+1}$ and in which $x_{n+1} = 1$. Then, by §3.7 we have $k_j(\mu) = \langle L(\mu), \alpha_{i_j}^{\vee} \rangle$ where $L : i\tilde{\mathfrak{t}}^* \longrightarrow i\mathfrak{t}^*$ is the projection which is dual to the composition $T \xrightarrow{\text{diag}} T^{(n+1)} \longrightarrow \tilde{T}$. In particular, k_j only depends on the integer i_j. When restricted to the subtorus T, our formula implies Demazure's formula.

Remark 3.23. Our formula, while refining the Demazure formula, bares a strong similarity to a formula of Littelmann and M. Kashiwara [L, K2] which we now describe. We use the notation of [K1, K2]. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra defined by Drinfeld and Jimbo. Let A be the subring of $\mathbb{Q}(q)$ which consists of the rational functions which are regular at $q = 0$. Let $V(\lambda)$ be the irreducible $U_q(\mathfrak{g})$-module of highest weight λ and let $(L(\lambda), B(\lambda))$ be its crystal base. Recall [K1], this means that $L(\lambda) \subset V(\lambda)$ is a free sub-A-module such that $V(\lambda) \cong \mathbb{Q}(q) \otimes_A L$, that $B(\lambda) \subset L(\lambda)/qL(\lambda)$ is a linear basis over $\mathbb{Q}$, and that the quantum operators $\tilde{e}_i$, $\tilde{f}_i : V(\lambda) \longrightarrow V(\lambda)$ send $L(\lambda)$ into itself. Let u_λ be the maximal weight vector in $V(\lambda)$. Let $w_0 = s_{i_1} \cdots s_{i_n}$ be a reduced expression for the maximal element w_0 in the Weyl group W. Then Littelmann's formula states that

$$\sum_{b \in B(\lambda)} b = \mathcal{D}_{i_1} \cdots \mathcal{D}_{i_n} u_\lambda$$

where $\mathcal{D}_i$ are defined in terms of $\tilde{e}_i$, $\tilde{f}_i$, by

$$\mathcal{D}_i b = \begin{cases} \sum_{0\le k\le \langle \alpha_i^\vee, weight(b)\rangle} \tilde{f}_i^k b & \text{if} \quad \langle \alpha_i^\vee, \text{weight}(b)\rangle \ge 0 \\ -\sum_{0<k<-\langle \alpha_i^\vee, weight(b)\rangle} \tilde{e}_i^k b & \text{if} \quad \langle \alpha_i^\vee, \text{weight}(b)\rangle < 0 \end{cases}$$

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