Families of Waveforms with good Auto and Cross Ambiguity Properties

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Abstract. We address the problem of constructing frequency hop pulse trains for use in multiuser radar and sonar systems. We suggest a new construction of a family of frequency hop waveforms with nearly ideal autoambiguity function and low crossambiguity properties across the entire family of waveforms.

The ambiguity function $A_f(\tau,\nu)$ ([7], [2]) of a transmitted signal $f(t)$ measures the uncertainty with which the returning echo distinguishes, simultaneously, both ranges and velocities of a target system. Generally speaking, $A_f(\tau,\nu)$ is desired to be of 'thumbtack' shape ([6]), i.e. a function whose absolute value has a graph with a strong peak at the origin over a broad shallow base.

On the other hand, as pointed out by Titlebaum, Maric and Bellegarda ([5]), if several active systems view the same target complex, signals from one system may be interpreted as echoes or outputs from the other system(s). A similar situation arises in asynchronous spread spectrum communications if crosstalk occurs between two or more of the signals (code words) considered. These interference problems are typical of such multiuser environments. To achieve jamming resistance or low probability of intercept, it is necessary to use a sequence of signals with small crossambiguity functions between any two elements of the sequence.

We use a well established tool, the Zak transform ([4],[1])

$$Z_f(x,y)=\sum_{k=-\infty}^{\infty} f(x+k)e^{2\pi iky}$$

to compute the crossambiguity function

$$A_{f,g}(\tau,\nu)=\int_{-\infty}^{+\infty} f(t)\overline{g(t-\tau)}e^{-2\pi it\nu}dt$$

$$=\int_0^1\int_0^1 Z_f(x,y)\overline{Z_g(x+\tau,y+\nu)}e^{-2\pi ix\nu}dx\,dy\,.$$

Since the ambiguity function can be computed from the Zak transform and because the ambiguity function has a simple form on the integer lattice in terms of the Zak transform, we will carry out the design of a family of signals $\{f_k\}$ with desirable auto and cross ambiguity properties in the Zak domain.

To illustrate the method, we describe one of several families of such designs, given by setting $Z_{f_k}(x,y)=\cos(2\pi k\|(x,y)\|_p)$ on $[0,1]\times[0,1]$, where $\|(x,y)\|_p=(x^p+y^p)^{1/p}$, and extending outside the unit square by the required quasiperiodic properties of the Zak transform. We defer for a moment a discussion of the determination of f_k's that will produce the above family of Zak transforms, and take up the autoambiguity function.

The autoambiguity function on the integer lattice is given by

$$A_{f_k}(n,m)=A_{f_k,f_k}(n,m)=$$

$$\int_0^1\int_0^1 |Z_{f_k}(x,y)|^2e^{2\pi i(-mx+ny)}dx\,dy\,,$$

so the $A_{f_k}(n,m)$'s are basically the Fourier coefficients of the real non-negative function $|Z_{f_k}(x,y)|^2$, and by cosine identities, they correspond in the above way to the Fourier coefficients of $\frac{1}{2}\cos(4\pi k\|(x,y)\|_p)+\frac{1}{2}$. A computation then shows that the autoambiguity function A_{f_k} is 'thumbtack' on the integer lattice, and it is also straightforward to see that the crossambiguity function A_{f_k,f_l} is small on the integer lattice.

We will illustrate an example of how one can design in the Zak domain a frequency hop pulse train

$$f(t)=\sum_{n=-N}^{N} a_n\chi_{[0,1]}(t-n)e^{2\pi i\theta_n t}\,,$$

where $\chi_{[0,1]}(t)$ is a pulse of duration 1 and $\theta_n\in\{-N,N\}$. Note that the corresponding Zak transform is given by

$$Z_{f_k}(x,y)=\sum_n a_ne^{2\pi i(\theta_n x+ny)}.$$

We begin the design by taking the Fourier development $\sum_{m,n=-\infty}^{+\infty} b_{mn}e^{2\pi i(mx+ny)}$ of $\cos(2\pi k\|(x,y)\|_p)$ on $[0,1]\times[0,1]$. Note that since $\cos(2\pi k\|(x,y)\|_p)$ is real,

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$b_{-m,-n} = b_{mn}$, and hence $|b_{-m,-n}| = |b_{mn}|$. Numerical evidence leads to the presence, for each n, of a unique m for which the corresponding $|b_{mn}|$ is a maximum. We will denote this maximum by a_n. We will denote the m-subscript corresponding to the maximum by θ_n. This generates the above pulse train, whose real and imaginary parts we illustrate in Fig. 1 for the case $k = 8$ and $p = 2.4$. Note that by the above, $a_{-n} = \overline{a_n}$ and $\theta_n = \theta_{-n}$.

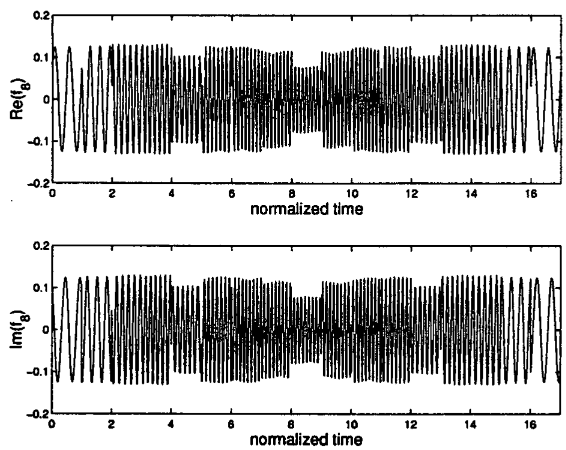

Fig. 1. Real and imaginary parts of the pulse train $f_8(t)$, p=2.4.

If we set

$$f_{k_1}(t) = \sum_{n=-k_1}^{k_1} a_n \chi_{[0,1]}(t-n) e^{2\pi i \theta_n t}$$

and

$$f_{k_2}(t) = \sum_{m=-k_2}^{k_2} b_m \chi_{[0,1]}(t-m) e^{2\pi i \theta_m t},$$

and write the crossambiguity $A_{f_{k_1}, f_{k_2}}(\tau, \nu)$ as the sum of $A^{(1)}(\tau,\nu)$ and $A^{(2)}(\tau,\nu)$, where

$$A^{(1)}(\tau,\nu) = \sum_n a_n \overline{b_n} e^{-2\pi i (n\nu - \theta_n \tau)} A_\chi(\tau,\nu),$$

$$A^{(2)}(\tau,\nu) =$$

$$\sum_{\substack{n,m \\ n \neq m}} a_n \overline{b_m} e^{-2\pi i (n\nu - \theta_m \tau)} A_\chi(\tau + (m-n), \nu + (\theta_m - \theta_n)),$$

we can easily see that the first term will produce the main lobe of the ambiguity function and the second term will produce the sidelobes. The location of the coefficients $\{a_n\}$ and $\{b_m\}$ in the matrices A and B will give rise to at most two hits between the matrix A and the shifted version of the matrix B for any two numbers k_1 and k_2. This implies that after the sum defining $A^{(2)}(\tau,\nu)$ is evaluated, the net result reduces to at most two easily identified terms of the original sum. Auto and cross ambiguity surfaces are illustrated in Fig. 2 and Fig. 3 respectively.

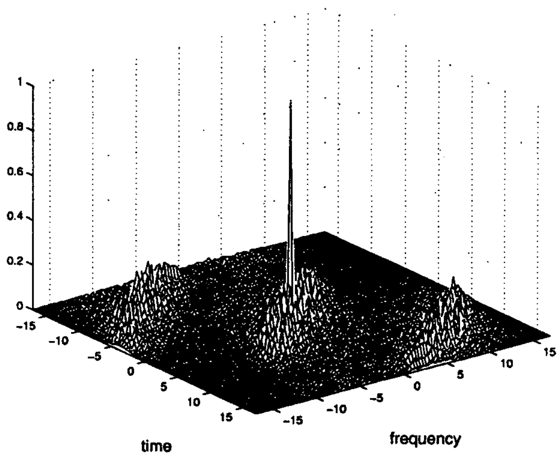

Fig. 2. Ambiguity surface $|A_{f_8}|$.

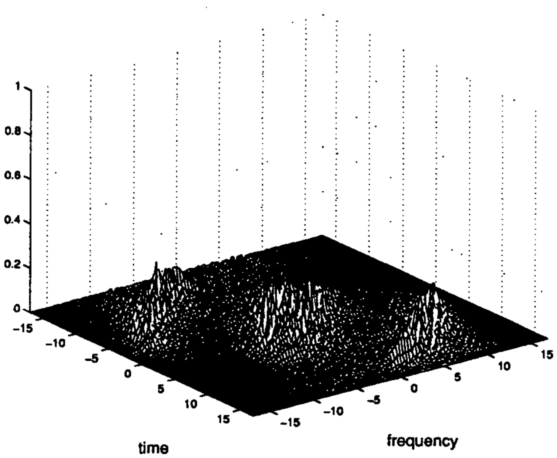

Fig. 3. Cross-Ambiguity surface $|A_{f_8,f_6}|$.

The symmetries among the coefficients suggest considering the real part $u_k(t)$ of our pulse train $f_k(t)$. After an easy computation, we find that the Zak transform corresponding to $u_k(t)$ is one half of

$$\sum_n a_n e^{2\pi i (\theta_n x + ny)} + \sum_n \overline{a_n} e^{2\pi i (-\theta_n x + ny)}.$$

By the properties of the a_n's and the θ_n's, this equals

$$\sum_n a_n e^{2\pi i(\theta_n x + ny)} + \sum_n a_n e^{2\pi i(\theta_n x + n(-y))}.$$

The first part of the last sum is an approximation to $\cos(2\pi k||(x,y)||_p)$ on $[0,1]\times[0,1]$, and therefore the second part of the sum is an approximation to $\cos(2\pi k||(x,y)||_p)$ on $[0,1]\times[-1,0]$. If we calculate the corresponding auto and cross ambiguity surfaces, we find that the sidelobes have been considerably spread out, and by the volume property of the ambiguity function, this results in overall lower sidelobe height. This is illustrated in Fig. 4 and Fig. 5.

To summarize, our method leads to

- Superior thumbtack properties of the ambiguity function, giving excellent performance in the presence of undesired time and frequency components.

- The construction of families of frequency hop pulse trains with the thumbtack property and good pairwise independence, as measured by cross ambiguity.

- Computationally simple signal designs.

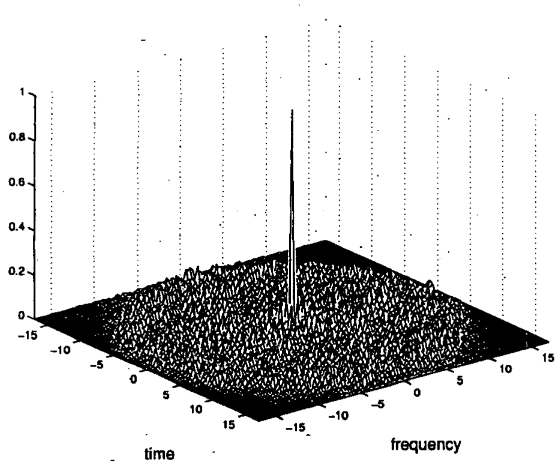

Fig. 4. Ambiguity surface $|A_{Re(f_8)}|$.

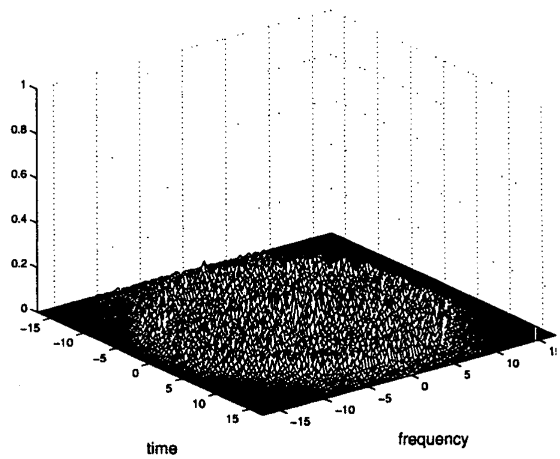

Fig. 5. Cross-Ambiguity surface $|A_{Re(f_8),Re(f_6)}|$.

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