On the synthesis problem for a waveform having a nearly ideal ambiguity surface*

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Abstract: In this paper we consider the problem of constructing a waveform with globally optimal ambiguity surface properties in a region surrounding the main lobe. We consider Hermite waveforms as the basis functions of our construction algorithm and discuss the problem of minimizing the volume under the ambiguity surface over a certain given region. In the case of a circular region, we show that under some rather general assumptions, a Hermite waveform of a certain order is a solution for this problem. We also consider an application of this approach to the design of bandpass signals with desired ambiguity profiles in the strip containing the time-delay axis.

Keywords: radar waveform design, ambiguity function, bandpass signals, Hermite waveform, nonlinear optimization.

1. Introduction

Waveform synthesis has been an important problem in radar design since the publication of Woodward's book [1], but despite numerous attempts to solve it, the search for practical solutions to the synthesis problem remains a challenging problem. The elegant paper of Wilcox [2] presents a mathematically complete solution, provided that the desired ambiguity shape is given in analytical form, which is not the case in any practical radar application. In practice, engineers have a general idea of acceptable shape rather than the formulas describing it, thus making Wilcox's algorithm not applicable. Moreover, in many situations, it is not even necessary to have a certain shape for all the values of time and doppler delays (the region where the ambiguity surface is desired to be small depends on the particular radar application), and Wilcox's algorithm does not treat the situations where only part of the ambiguity surface has to be approximated.

We have made an attempt in [3] to extend Wilcox's classical results to the case of subregions of R^2 and have shown that this generalization enables us to construct many promising new waveforms with desired ambiguity profiles in the regions surrounding the main lobe. In this paper we continue this work and consider an application of this approach to the design of bandpass signals with desired ambiguity profiles in the strip containing the time-delay axis.

2. Radar waveforms and their ambiguity functions

While considering a radar waveform, it is natural to describe it by a square-integrable and normalized function of time $u(t)$ that has finite timewidth and bandwidth, i.e. $u(t), tu(t)$, and $fU(f)$ are all in L^2_R and $\|u(t)\|_{L^2_R} = 1$ (here $U(f)$ denotes the Fourier transform of $u(t)$). We will denote a class of functions satisfying the above properties by W.

For each waveform $u(t) \in W$ we define its ambiguity function by

$$\chi_u(\tau,\nu) = \int_{-\infty}^{\infty} u\left(t - \frac{\tau}{2}\right) \overline{u\left(t + \frac{\tau}{2}\right)}\, e^{-j2\pi\nu t}\, dt$$

The ambiguity function as well as the class W itself have many remarkable properties (see [1, 4, 5], etc). In this section we recall some of the fundamental results needed in our work that first were obtained in the well-known paper by Wilcox [2].

Let us consider a subclass $W_0 \subset W$ of waveforms having their epoch and carrier frequencies both equal to zero, that is $< tu, u >_{L^2_R} = < fU, U >_{L^2_R} = 0$, where

$$< g, h >_{L^2_E} = \int_E g\, \overline{h}\, dE, \qquad \|g\|^2_{L^2_E} = < g, g >_{L^2_E} .$$

It is possible to select a sequence of members of W_0

$$\phi_0(t), \quad \phi_1(t), \quad \ldots, \quad \phi_m(t), \quad \ldots \tag{1}$$

which is complete and orthonormal in the mean square sense. Then, by the Riesz-Fischer theorem [6], each $u(t) \in W$ can be represented as

$$u(t) = \lim_{N\to\infty} \sum_{m=0}^{N} a_m \phi_m(t), \tag{2}$$

where $a_m = < u, \phi_m >_{L^2_R}$ and $\lim\limits_{N\to\infty} \sum\limits_{m=0}^{N} |a_m|^2 = 1.$

Sequence (1) induces the sequence of the cross-ambiguity functions

$$\psi_{mn}(\tau,\nu) = \int_{-\infty}^{\infty} \phi_m\left(t - \frac{\tau}{2}\right) \overline{\phi_n\left(t + \frac{\tau}{2}\right)}\, e^{-j2\pi\nu t}\, dt$$

which is also known to be orthonormal and complete in $L^2_{R^2}$. Hence, every ambiguity function $\chi_u(\tau,\nu)$ can be expanded in a series

$$\chi_u(\tau,\nu) = \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{m=0}^{N} b_{km} \psi_{km}(\tau,\nu)$$

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with $b_{km} =< \chi_u, \psi_{km} >_{L^2_{R^2}}$, $\lim_{N\to\infty} \sum_{k=0}^{N} \sum_{m=0}^{N} |b_{km}|^2 = 1$. Hermite waveforms defined as follows

$$u_n(t) = \frac{2^{1/4}}{\sqrt{n!}} H_n(2\sqrt{\pi}\, t) e^{-\pi t^2}, \quad n = 0, 1, 2, \ldots, \tag{3}$$

where $H_n(x)$ is the nth Hermite polynomial [2, 7], provide a well-known and quite interesting example of the basis (1). The ambiguity functions corresponding to the $u_n(t)$ can be expressed by

$$A_n(\tau, \nu) = e^{-\pi(\tau^2+\nu^2)/2} L_n(\pi(\tau^2 + \nu^2)), \tag{4}$$

where $L_n(x)$ is the nth Laguerre polynomial. Since Hermite waveforms give rise to a complete basis in L^2_R, any waveform can be represented as an infinite linear combination (2). Fig. 1 shows the first 1000 coefficients a_m's of this decomposition for the chirp signal $\mathrm{rect}(t - \frac{1}{2}) e^{j\, 2\pi(100t^2 - 100t)}$. (To be exact, figure 1 displays only coefficients with odd indices, since the coefficients with even indices are all equal to zero due to some properties of Hermite waveforms). As can be seen from the figure, the coefficients with $m > 500$ are small in absolute value and can be neglected in (2).

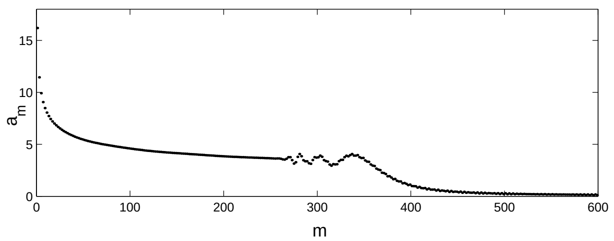

Figure 1: First 1000 coefficients a_m of the decomposition (2) for chirp waveform

Thus, the infinite series (2) that is equal to the chirp waveform can be replaced by a finite sum of the first 500 terms, giving us a very good approximation of the waveform as well as its ambiguity function. This observation will be used in the formulation of the optimization problem in the following section.

3. Optimization Problem

The ideal radar waveform would produce an ambiguity function that is zero everywhere except the origin. Such a function (we will denote it by $\chi_{opt}(\tau, \nu)$) would have ideal range-doppler characteristics. However, since no finite energy signal gives rise to that surface [5, 8], of ideal ambiguity surface somehow in order to get realizable waveforms with some optimal properties. For many purposes it is desirable to construct a waveform producing a surface which is very small everywhere in some (perhaps, quite large) neighborhood of the origin and has a peak at that point. Therefore, a practical version of the problem can be stated as follows:

Find a waveform $u(t) \in W$ such that its ambiguity surface $|\chi_u(\tau, \nu)|$ is the best approximation to the function $\chi_{opt}(\tau, \nu)$ in the mean square sense over some bounded 1-connected region G containing the origin.

In other words, we look for $u(t)$ that minimizes $\|\chi_{opt}(\tau,\nu) - |\chi_u(\tau,\nu)|\|^2_{L^2_G} = \|\chi_u(\tau,\nu)\|^2_{L^2_G}$. A variant of the above problem we discuss in this paper arises when we restrict $u(t)$ to a finite dimensional space. Namely, assuming that some orthonormal basis (1) is fixed, we define the sequence of classes W_N $(N = 0, 1, \ldots)$ as follows

DEFINITION. A function $u(t)$ is in class W_N $(N = 0, 1, \ldots)$ $\iff$ $u(t) = \sum_{m=0}^{N} a_m \phi_m(t)$ such that $a_m =< u, \phi_m >_{L^2_R}$ and $a_m \in S_N$, where S_N is the N-dimensional unit sphere: $\sum_{m=0}^{N} |a_m|^2 = 1$.

It is easy to see that the following lemma holds.

LEMMA 1. 1) $W_0 \subset W_1 \subset W_2 \subset \ldots$; 2) $W_N \subset W, \forall\, N$.

We should note that by considering the space W_N instead of W we are restricting our selection of possible solutions for the problem of minimizing the ambiguity surface in some given region G. Nevertheless, as it was shown in the previous section, W_{500} contains a good approximation of the chirp waveform with almost identical ambiguity functions. This leads to the *final* modification of the optimization problem:

Find $u(t)$ from the class W_N with minimal ambiguity function in the least square sense in some given region G, i.e.

$$\underset{u(t)\in W_N}{\arg\min} \|\chi_u(\tau, \nu)\|^2_{L^2_G}. \tag{5}$$

Finally, we remark that it can be shown that any solution $u^{(N)}_*(t)$ of problem (5) is also a solution of $\underset{u(t)\in W_N}{\arg\min} \|\chi_u(\tau,\nu)\|^2_{L^1_G}$ which simply means that the volume under $|\chi_{u^{(N)}_*}|$ over G is the smallest one compared with the ambiguity surfaces produced by the other members of W_N.

4. Optimal solution in W_N

Note that since, for any $u(t) \in W_N$, its ambiguity function admits a representation of the form

$$\chi_u(\tau, \nu) = \sum_{i=0}^{N} \sum_{k=0}^{N} a_i \bar{a}_k \psi_{ik}(\tau, \nu),$$

one can write

$$\|\chi_u(\tau, \nu)\|^2_{L^2_G} = Q(\mathbf{a}) = \sum_{i,k,m,n=0}^{N} c_{ikmn} a_i \bar{a}_k \bar{a}_m a_n$$

where $c_{ikmn} =< \psi_{ik}(\tau, \nu), \psi_{mn}(\tau, \nu) >_{L^2_G}$. Thus, we conclude that to solve problem (5) we have to *find $a_m \in S_N$ $(m = 0, 1, \ldots, N)$ minimizing the 4th order form $Q(\mathbf{a})$*. At this point we restrict ourselves by considering problem (5) for the case when the basis functions in (1) are Hermite waveforms (3). The following theorem states a very interesting optimal property of Hermite waveforms:

THEOREM. Let $r > 0$ be some arbitrarily chosen real number and G be the circular region: $G = \{(\tau, \nu) : \tau^2 + \nu^2 \leq r\}$. Let also N be a fixed nonnegative integer. Then the Nth Hermite waveform (3) is a solution of problem (5) among all the waveforms from W_N.

The PROOF of the theorem is omitted here for brevity and can be found in full in [9].

The waveform $u_{20}(t)$ is depicted in fig. 2 (top) and a cross-section of the corresponding ambiguity surface is presented in fig. 2 (bottom).

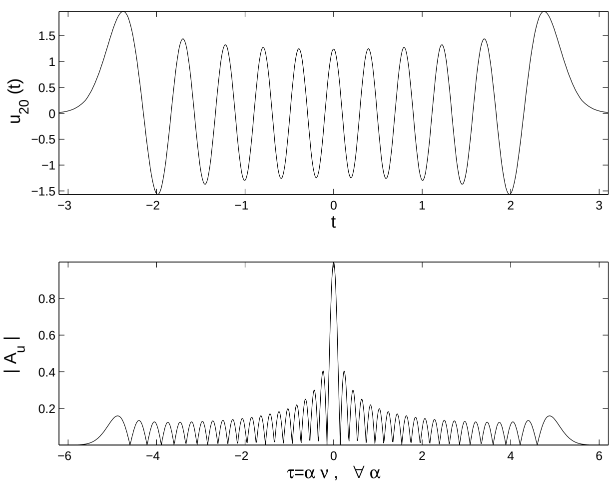

Figure 2: Hermite waveform $u_{20}(t)$ and arbitrary cross-section passing through the origin for its ambiguity surface

The ambiguity surface produced by a Hermite waveform has circular symmetry (see (4)), that is, its graph is a surface of revolution. So, to imagine the entire surface it is enough to know only one of the cross-sections passing through the origin. All others are exactly the same.

It follows from the above theorem that the ambiguity surface of the Nth Hermite waveform provides the best (among all the members of W_N) approximation of the ideal ambiguity surface in the mean square sense over *any* circular region centered at the origin. Therefore, the larger N we will choose, the better approximation of the ideal shape we will obtain (this fact is illustrated in fig. 2 and 3).

We should note at this point, that Hermite waveforms and their other various optimal properties have been known in the radar community for a long period of time [2, 7], but have not been extensively used for practical needs of radar design. One of the reasons is that the few sidelobes (for each of the signal from this family) are relatively high and their level does not become lower with an increase of N. To illustrate this phenomenon, we have displayed in fig. 4 the cross-section of the ambiguity surface in the vicinity of the main lobe for four different Hermite waveforms. As it can be seen from the figure, when N increases, the kth sidelobe $(k = 1, 2, \ldots)$ of the Hermite ambiguity surface approaches the origin, becomes more narrow, and, unfortunately, preserves the same height. This property holds for all Hermite waveforms, not only for those depicted in fig. 4, thus making the Hermite family of waveforms not suitable for most applications.

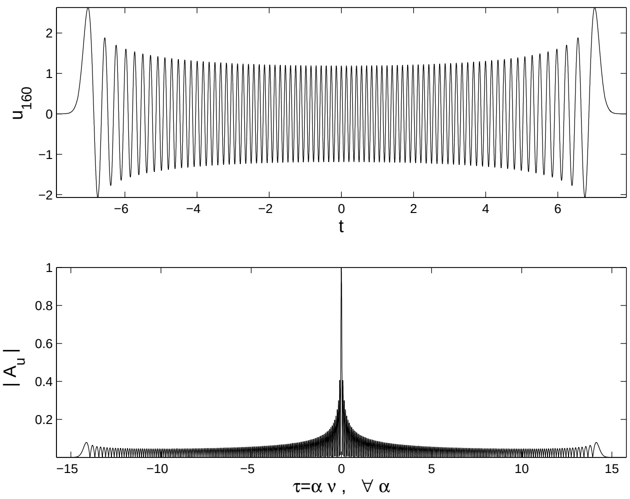

Figure 3: Hermite waveform $u_{160}(t)$ and arbitrary cross-section passing through the origin for its ambiguity surface

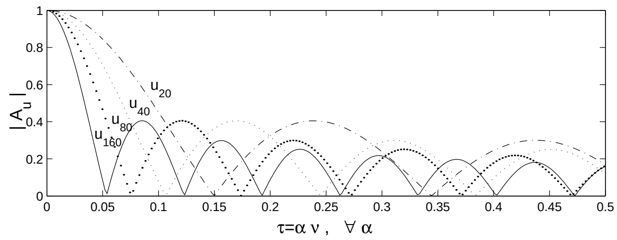

Figure 4: Sidelobe level of ambiguity surface for different Hermite waveforms

We should remark that the sidelobe problem is unfortunate but not surprising, since the optimization problem (5) has been formulated in terms of minimal volume under the surface. Reformulation of the problem (5) in terms of the max-norm (or L^{∞}_{G}-norm) that would guarantee the lowest level of sidelobes in the selected region G is, obviously, one way to address the issue of high sidelobes, but in this case the problem becomes more complicated and remains open.

However, we have recently shown in [3], that the replacement of the selected region G, rather than norm (or basis functions) can lead to acceptable solutions. In [3] we consider the minimization problem (5) over a circular ring $G_0 = \{(\tau, \nu) : r_0^2 \leq \tau^2 + \nu^2 \leq r^2\}$ and show that appropriate choice of the inner radius r_0 enables us to find a solution of (5) with low sidelobe level of its ambiguity shape not only over G_0, but for the whole G. We should also note that the resulting waveforms as well as their ambiguity functions depend significantly on N and r and the question regarding a general solution of this problem remains open.

5. Applications to bandpass signals

In this section we discuss one of the possible applications of the theory presented in this paper for design of radar waveforms with acceptable range-doppler characteristics.

We will discuss in what follows the construction of bandpass signal that has good ambiguity profiles along the time-delay axis, i.e. when the region G of interest is chosen as $G_T = \{(\tau, \nu) : |\tau| \le T, |\nu| \le \nu_*\}$ and the signal $s(t)$ is of constant amplitude and finite support, i.e.

$$s(t) = \text{rect}\left(\frac{t}{T}\right) e^{j\pi\theta(t)}, \qquad (6)$$

where T is a signal's duration and $\theta(t)$ is its phase. We will assume here that the phase function $\theta(t)$ satisfies conditions that allow $s(t)$ to belong to class W.

The parameter ν_* that determines the width of the strip G_T and the time duration parameter T of the signal $s(t)$ should be selected depending on the application of interest. Both of these parameters, obviously, have an effect on the solution(s) of the problem (5) as well as on the sidelobe levels of the resulting ambiguity surface.

It is evident that the class W contains signals of both constant and nonconstant amplitude, while any of its subclasses W_N consists of only waveforms with varying amplitude. Although our theory is developed to work with finite-dimensional subclasses of W which do not contain bandpass signals, it was mentioned earlier and illustrated for the case of the chirp waveform that W_N contain acceptable approximations of signals with constant amplitude. Therefore, we will restrict the class W_N to a subset of signals with envelopes that are 'close' to rect$\,(t/T)$ and conduct the search for the solution of problem (5) subject to these additional constrains. The facts that are necessary for construction of this subclass are given by the following statements which we mention here without proof (see [9] for details).

LEMMA 2. The kth ($k = 0, 1, 2, \ldots$) time moment of a signal (6) equals $2b_k$, where

$$b_k = \begin{cases} (T/2)^{k+1}/(k+1), & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

LEMMA 3. If $u(t) \in W$ has a kth time moment, then

$$\int_{-\infty}^{\infty} t^k |u(t)|^2 dt = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn}^{(k)} Re\{a_m \overline{a}_n\}, \qquad (7)$$

where a_i's are the coefficients of expansion (2) over the basis of Hermite waveforms and

$$c_{mn}^{(k)} = \begin{cases} \frac{\sqrt{(n+1)(n+2)\ldots(n+k)}}{(2\sqrt{\pi})^k}, & \text{if } m = n + k, \\ 0, & \text{otherwise.} \end{cases}$$

From lemmas 2 and 3 we can derive the following

COROLLARY. If $s(t) \in W$ is a bandpass signal and $\{a_i\}$ are its coefficients of expansion (2) over basis (3), then, for $k = 1, 2, \ldots$,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn}^{(k)} Re\{a_m \overline{a}_n\} = 2b_k. \qquad (8)$$

We should remark that for the finite-dimensional case W_N, formula (7) becomes

$$\int_{-\infty}^{\infty} t^k |u(t)|^2 dt \approx 2 \sum_{m=0}^{N} \sum_{n=0}^{m} c_{mn}^{(k)} Re\{a_m \overline{a}_n\}.$$

and the system of equations (8) should be replaced by the following system of inequalities

$$\left| \sum_{m=0}^{N} \sum_{n=0}^{m} c_{mn}^{(k)} Re\{a_m \overline{a}_n\} - b_k \right| < \varepsilon_k, \qquad (9)$$

Now we define a class $\widetilde{W}_N$ to be a subset of W_N consisting of waveforms satisfying the system of constrains (9). We use $\widetilde{W}_N$ as our new feasible set in problem (5) performing optimization over G_T. Note that the way we have defined the class $\widetilde{W}_N$ does not restrict all its elements to the projection of a set of bandpass signals onto W_N. By (9), we restrict our attention only to members of W_N whose envelopes are "relatively close" to the shape of a rectangular pulse with predefined time support. Numerical analysis we have conducted shows that, for large values of N, the class $\widetilde{W}_N$ contains a variety of waveforms with excellent ambiguity profiles in the region of interest. Once such a waveform is found, we force it to be of a constant amplitude with the same phase which is justified by the following lemma.

LEMMA 4. Let $s_1(t)$ and $s_2(t)$ be the members of the class W such that $s_1(t)$ is represented by (6) and

$$s_2(t) = \left(\text{rect}\left(\frac{t}{T}\right) + \mu(t)\right) e^{j\pi\theta(t)},$$

where $\|\mu(t)\|_{L^2_R} \le \varepsilon$, $\varepsilon > 0$ is a small real number. Then,

$$\|\, |\chi_{s_1}(\tau,\nu)| - |\chi_{s_2}(\tau,\nu)|\, \|_{L^2_{G_T}} \le 2\varepsilon\sqrt{T\nu_*}(2\sqrt{T} + \varepsilon).$$

Lemma 4 states that if both $s_1(t)$ and $s_2(t)$ have the same phase and their amplitudes are similar in the mean square sense, we should not expect much of a difference between their ambiguity profiles. Therefore, due to Lemma 4 and the definition of the class $\widetilde{W}_N$, forcing of the envelope to be constant in our case does not lead to essential degradation of ambiguity profiles in doppler-delay plane.

Figures 5-8 illustrate the procedure described above, where $s^*(t)$ is the numerical solution of problem (5) in the subspace $\widetilde{W}_{1000}$ over region $G_T = \{(\tau, \nu) : |\tau| \le T, |\nu| \le 1/(8T)\}$.

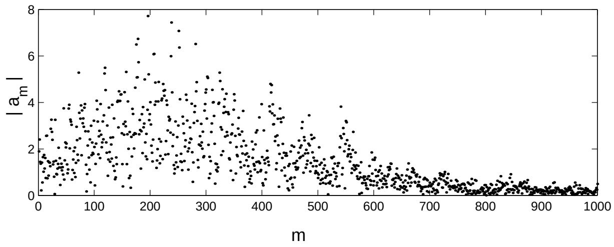

Figure 5: Coefficients a_m of the signal $s^*(t)$ from $\widetilde{W}_{1000}$

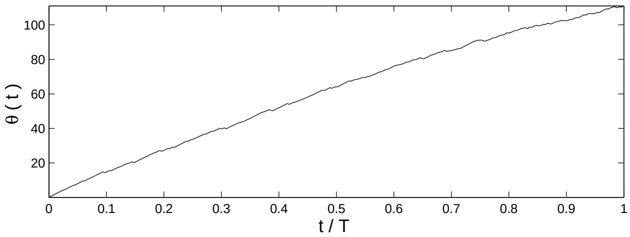

Figure 6: Phase $\theta(t)$ (bottom) of the signal $s^*(t)$ from $\widetilde{W}_{1000}$

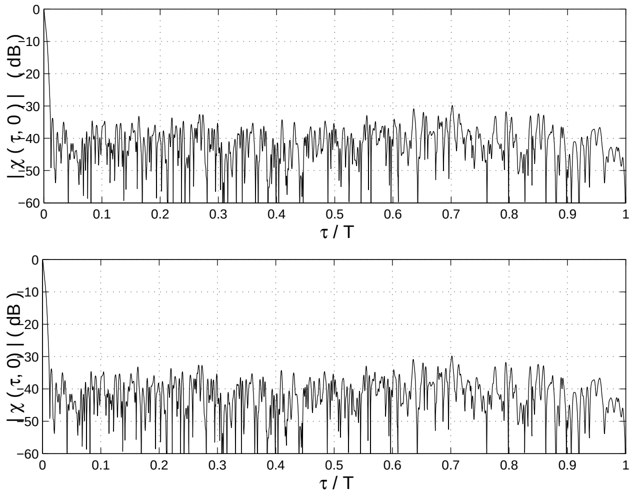

Figure 7: Autocorrelation functions of both the (amplitude modulated) signal $s^*(t)$ from $\widetilde{W}_{1000}$ (top) and the corresponding bandpass signal (bottom)

6. Conclusion

In this paper we discuss a new approach to one of the most challenging problems of the radar waveform design - the construction of waveforms with optimal ambiguity characteristics in a chosen a priori region surrounding the main lobe. Our approach is based on the projection of the signal onto an appropriate orthonormal basis in the space of radar waveforms and approximating the signal with desired ambiguity properties by a finite number of basis waveforms. In this paper we consider the well-known Hermite waveforms as the basis functions and discuss the problem of minimizing the volume under the ambiguity surface over a certain given region. In the case of a circular region, we show that under some rather general assumptions, a Hermite waveform of a certain order is a solution for this problem. We also consider an application of this approach to the design of bandpass signals with desired ambiguity profiles in the strip containing the time-delay axis.

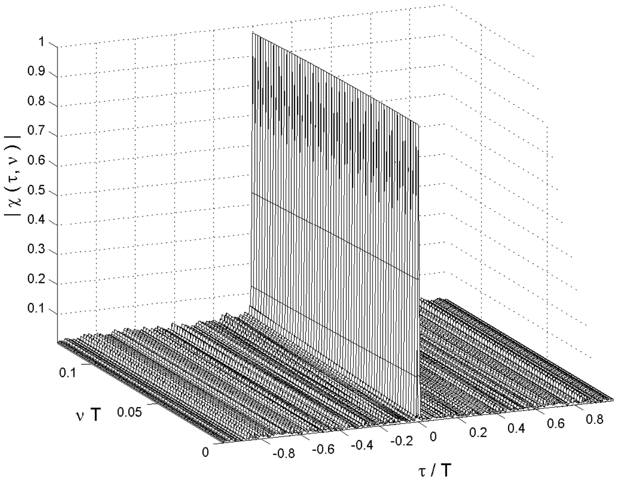

Figure 8: Partial ambiguity plot of the bandpass signal corresponding to $s^*(t)$ (zoom on G_T)

Finally, we should remark that this work (along with [3] and [9]) is only a beginning and should be carried out further to develop general algorithms for the construction of realizable waveforms with optimal characteristics. In particular, some of the remaining issues outlined in this paper are difficult and remain open.

7. Acknowledgment

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8. References

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