To the approximation of ideal ambiguity surface

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Abstract

In this paper, we extend our modification of the waveform approximation technique proposed by Wilcox (1960) to the case of the cross-ambiguity surface. This modification allows the design of a non-linear frequency modulated signal to be transmitted by radar and a corresponding reference signal to be used by the matched filter. This pair is designed so that the sidelobes of the cross-ambiguity surface are minimal in a specified region of interest.

1 Introduction

The ambiguity function of a waveform $u(t)$ is defined by (Woodward, 1953)

$$\chi_u(\tau, \nu) = \int_{-\infty}^{\infty} u(t)\, \overline{u(t-\tau)}\, e^{-j2\pi\nu t}\, dt, \tag{1.1}$$

where $u(t)$ is assumed to be a function of time with unit energy. A radar waveform with ideal range-doppler characteristics would produce an ambiguity surface that is zero everywhere except the origin. However, no finite energy signal gives rise to such a surface (Cook & Bernfeld, 1967). Waveform synthesis has been an important problem in radar design since the publication of the book (Woodward, 1953), but despite numerous attempts to solve it, the search for practical solutions to the synthesis problem remains open. The fundamental paper of Wilcox (1960) presents a mathematically complete solution (using Hilbert space techniques), provided that the desired ambiguity shape is given in analytical form, which is not the case in any practical radar applications. In previous papers (Gladkova & Chebanov, 2004a-c) we have adapted Wilcox's method to the case of specified subregion of R^2 and have shown that this generalization enables us to construct many promising new waveforms with desired ambiguity profiles in the regions surrounding the main lobe.

The optimization over a subregion of R^2 generalizes Wilcox's approach, which optimizes over all of R^2. There are new subtleties that appear with this approach, since we can seek, for example, to make an ambiguity small over some region, which, if successful, will push the bulk of the function outside the region where we want it to be small. Obviously, this is not possible if the region is all of R^2, because of the volume property of the ambiguity function.

We should note that one of the desired features of radar waveforms is constant amplitude (due to some radar hardware limitations). This greatly complicates the design of waveforms with prescribed ambiguity surfaces. One of the possibilities that allows the consideration of waveforms with variable amplitude is to work with a pair of waveforms (Levanon & Mozeson, 2004): the transmitted signal of constant amplitude and the reference signal of arbitrary amplitude that is used during the signal processing stage at the receiver. Thus we are interested in cross-ambiguity function

$$\chi_{u_{\text{tr}}, u_{\text{ref}}}(\tau, \nu) = \int_{-\infty}^{\infty} u_{\text{tr}}(t)\, \overline{u_{\text{ref}}(t-\tau)}\, e^{-j2\pi\nu t}\, dt, \tag{1.2}$$

where $u_{\text{tr}}(t) = u_0 e^{j\pi w(t)}$ is a transmitted frequency modulated signal and $u_{\text{ref}}(t) = \sigma(t) e^{j\pi w(t)}$ is a reference signal with varying envelope $\sigma(t)$.

The focus of this paper is to consider the problem of sidelobe suppression of the cross-ambiguity surface over the given region of interest. Thus we extend our approach introduced in (Gladkova & Chebanov, 2004a-c) to the case of designing a pair of transmitted/reference waveforms with desired characteristics.

2 Optimization Problem

Since any waveform $u(t)$ under consideration is a square-integrable function of time (that is $u(t) \in L^2_R$), it can be represented, by the Riesz-Fischer theorem (Riesz & Sz.-Nagy, 1955), as

$$u(t) = \lim_{N\to\infty} \sum_{m=0}^{N} a_m \phi_m(t), \tag{2.1}$$

where $a_m =< u, \phi_m >_{L^2_R}$, $\lim\limits_{N\to\infty} \sum\limits_{m=0}^{N} |a_m|^2 = 1$, and the sequence

$$\phi_0(t), \quad \phi_1(t), \quad \ldots, \quad \phi_m(t), \quad \ldots \tag{2.2}$$

constitutes an orthonormal basis in L^2_R. In the above formulas and hereafter, we use the following notations:

$$< g, h >_{L^2_E} = \int_E g\,\overline{h}\,dE, \qquad \|g\|^2_{L^2_E} =< g, g >_{L^2_E},$$

where $g, h \in L^2_{R^k}$ and $E \subseteq R^k$ for some positive integer k.

Assuming that some orthonormal basis (2.2) is fixed, our further consideration will be related to the class V_N defined as follows:

Definition 1. A function $u(t)$ is in class $V_N \iff u(t) = \sum\limits_{m=0}^{N} a_m \phi_m(t)$, such that $a_m =< u, \phi_m >_{L^2_R}$ and $a_m \in S_N$, where S_N is the N-dimensional unit sphere: $\sum\limits_{m=0}^{N} |a_m|^2 = 1$.

Now, we can formulate the problem of finding $u_{\text{ref}} \in V_N$, corresponding to a given transmitted signal $u_{\text{tr}}(t)$, so that the cross-ambiguity surface $|\chi_{u_{\text{tr}},u_{\text{ref}}}(\tau,\nu)|$ has the prescribed shape in the given region G. From a practical point of view, it is desirable to construct waveforms producing surfaces which are very small everywhere in some (perhaps, quite large) neighborhood of the origin and have a peak at that point. Based on this observation, we can formulate a waveform design problem as follows:

Find a waveform $u_{\text{ref}}(t) \in V_N$ such that the cross-ambiguity surface $|\chi_{u_{\text{tr}},u_{\text{ref}}}(\tau,\nu)|$ is the best approximation to the ideal ambiguity surface in the mean square sense over some bounded region G containing the origin, i.e.

$$\arg\min_{u_{\text{ref}} \in V_N} \|\chi_{u_{\text{tr}},u_{\text{ref}}}\|^2_{L^2_G} \tag{2.3}$$

It should be noted here that the solution(s) of the non-linear problem (2.3) significantly depend on the choice of the region G as well as basis functions $\{\phi_k(t)\}$. In this paper we consider the orthonormal basis which is known in the literature as the most energy concentrated basis in the space of bandlimited signals.

3 Prolate spheroidal functions

We now specialize to the case where the orthonormal basis (2.2) consists of prolate spheroidal wave functions. The elegant standard reference for these functions is (Slepian, 1983), from which we now recall their definition and a few basic properties.

Consider the following optimization problem:

$$\text{maximize} \;\; \frac{\int_{-T/2}^{T/2} u^2(t)\,dt}{\int_{-\infty}^{\infty} u^2(t)\,dt}, \tag{3.1}$$

for all functions in L^2_R whose amplitude spectra vanish for $|\nu| > W$, i.e. in the space of bandlimited signals B_W.

Solutions (prolate spheroidal functions) of (3.1) satisfy an integral equation with kernel $\sin(\pi WT(t-t'))/(\pi(t-t'))$, i.e.

$$\int_{-1}^{1} \frac{\sin(\pi WT(t-t'))}{\pi(t-t')} \psi(t')\,dt' = \lambda\psi(t), \;\; |t| > 1 \tag{3.2}$$

and also satisfy a second order linear ordinary differential equation

$$\frac{d}{dt}(1-t^2)\frac{d\psi}{dt} + (\chi - (\pi WT)^2 t^2)\psi = 0. \tag{3.3}$$

The symmetric kernel of (3.2) is positive definite, therefore (3.2) has solutions in $L^2_{(-1,1)}$ only for a discrete set of real positive values of λ, which we will denote $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots$, with the corresponding eigenfunctions $\psi_0(t)$, $\psi_1(t)$, $\psi_2(t), \cdots$ that can be chosen to be real and orthogonal on $(-1, 1)$. They are also complete in $L^2_{(-1,1)}$. The left hand side of (3.2) is well defined for all $t \in R$, so $\psi_n(t)$ can be defined on R and normalized to unit energy there. Then λ_n is the fraction of the energy of ψ_n that lies in the interval $(-1, 1)$. Prolate spheroidal wave functions have many remarkable properties and can provide a very useful set of bandlimited signals that can be defined as follows (Slepian, 1983):

Suppose $W > 0$ and $T > 0$ are given. Define $\phi_n(t) = \sqrt{\frac{2}{T}}\ \psi_n\left(\frac{2t}{T}\right)$.

Then

- $\phi_n(t) \in B_W$
- $\int\limits_{-T/2}^{T/2} \phi_n(t)\phi_m(t)dt = \lambda_n\delta_{mn}$
- $\int\limits_{-\infty}^{\infty} \phi_n(t)\phi_m(t)\,dt = \delta_{mn}$
- the $\phi_n(t)$ are complete in B_W for $t \in R$
- the $\phi_n(t)$ are complete in $L^2_{(-T/2,T/2)}$
- among signals in B_W, $\phi_n(t)$ is the most concentrated signal that is orthogonal to $\phi_0(t), \phi_1(t), \ldots, \phi_{n-1}(t)$

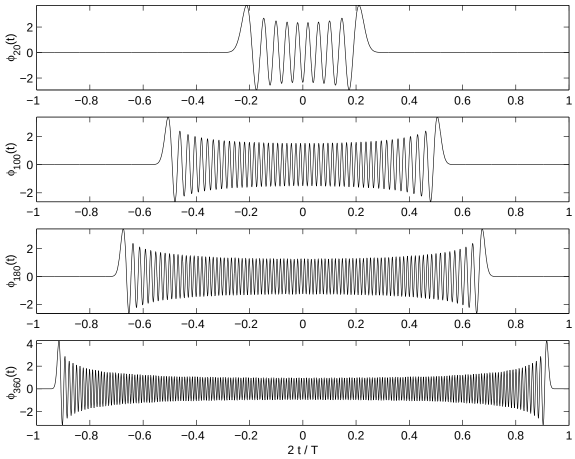

Figure 1: $\phi_n(t)$ for some values of n.

4 Numerical results

In what follows, our calculations with prolate spheroidal wave functions will be carried using Matlab's Signal Processing Toolbox program dpss and the minimization problem (2.3) is solved numerically using Matlab's Optimization Toolbox function fmincon. Next we will consider a numerical solutions of the minimization problem (2.3) for the special case when the region G is $[-T, -\varepsilon T] \cup [\varepsilon T, T] \times [-1/T, 1/T]$, i.e. a union of two strips along the time-delay axis that does not include an ε neighborhood of the main lobe. The choice of ε is not arbitrary, it affects the solution of the optimization problem (in the example considered below ε was chosen to be 1/256). The linear frequency modulated waveform (chirp) was chosen as the initial approximation of the transmitted signal $u_{\text{tr}}(t)$.

Figure 2 illustrates the ambiguity surface in the vicinity of the main lobe. The cross-section in logarithmic scale is depicted in figure 3. The height of the main lobe is 0.8233 and all sidelobes in the considered region G are below -42dB. The phase $w(t)$ and the envelope $\sigma(t)$ are illustrated in the figure 4.

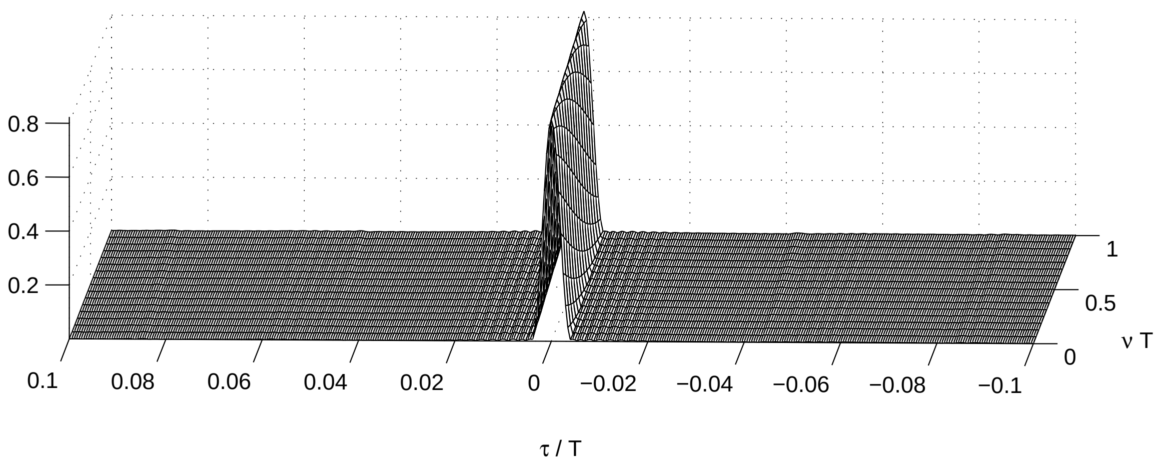

Figure 2: Partial cross-ambiguity surface with suppressed sidelobes.

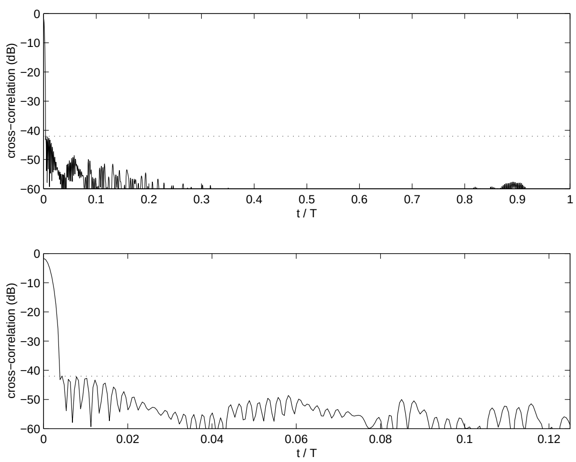

Figure 3: Autocorrelation function (in dB) for $0 < \tau < T$ (top) and $0 < \tau < T/8$ (bottom)

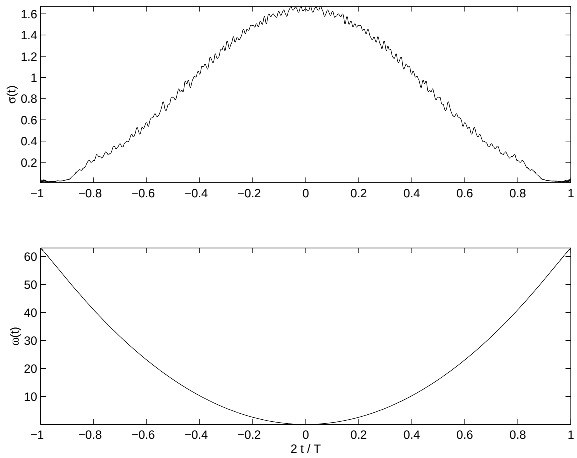

Figure 4: Envelope $\sigma(t)$ (top) and phase $w(t)$ (bottom) of waveforms $u_{\text{tr}}(t) = u_0 e^{j\pi w(t)}$ and $u_{\text{ref}}(t) = \sigma(t)e^{j\pi w(t)}$.

Conclusion

The inverse problem of finding a waveform corresponding to the prescribed ambiguity surface is one of the most challenging problems remaining open for over 5 decades. It is even more complicated if one wants to find a practical solution imposed by some hardware limitations. For example when signal desired to be of constant amplitude, compactly supported, bandlimited, and with low sidelobes of the ambiguiy surface. Such signals do not exist and therefore some trade-offs are inevitable. One of the possible trade-offs is to consider a pair of signals (instead of one), a transmitted and reference signal, and a part of the cross-ambiguity surface which depends on the particlar application (rather than the whole surface). This formulation is a focus of our paper and we have presented an approach that is based on the projection of the transmitted signal onto an appropriate orthonormal basis and approximating the reference signal with desired cross-ambiguity surface by a finite number of basis waveforms.

Acknowledgments

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