Sampling Theory

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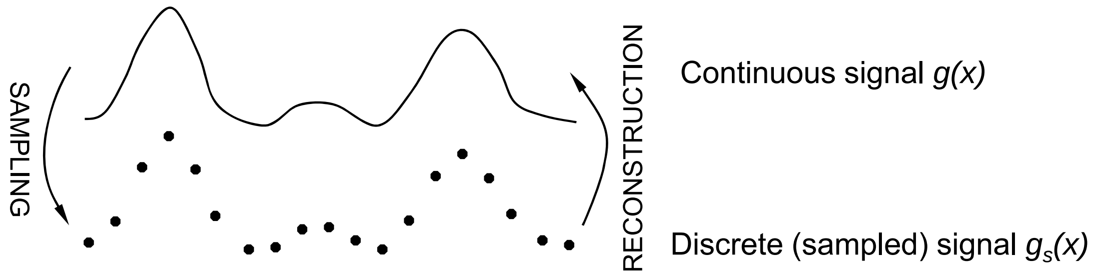
Objectives

- In this lecture we describe sampling theory:
  - Motivation
  - Analysis in the spatial and frequency domains
  - Ideal and nonideal solutions
  - Aliasing and antialiasing
  - Example: oversampling CD players

Sampling Theory

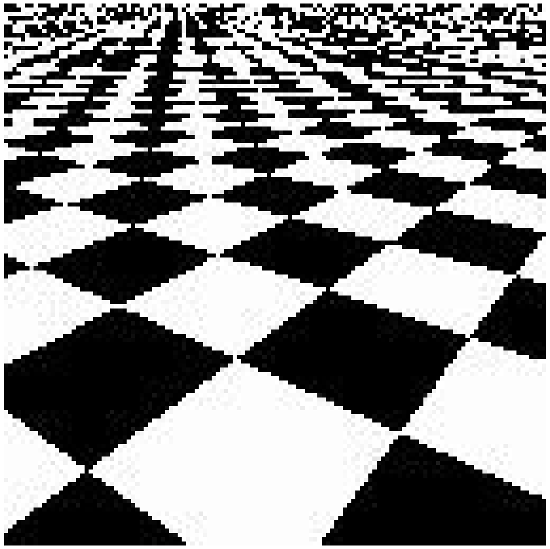


Sampling theory addresses the two following problems:

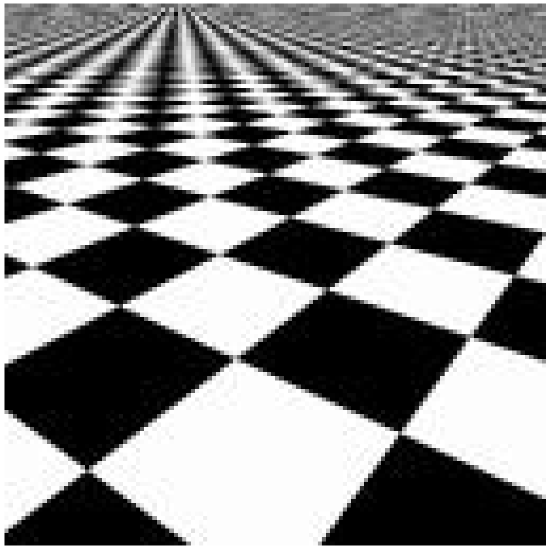
1. Are the samples of $g_s(x)$ sufficient to exactly describe $g(x)$?
2. If so, how can $g(x)$ be reconstructed from $g_s(x)$?

Motivation

- Improper consideration leads to jagged edges in magnification and Moire effects in minification.

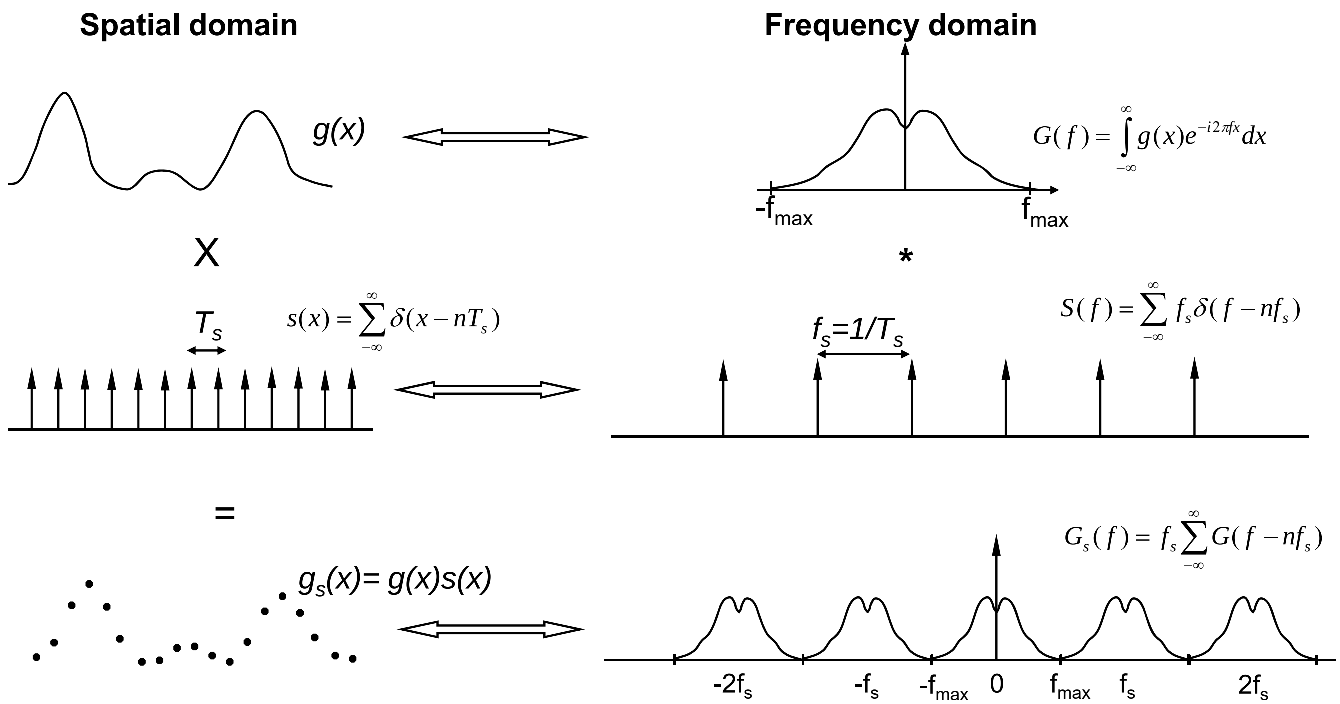


Unfiltered image



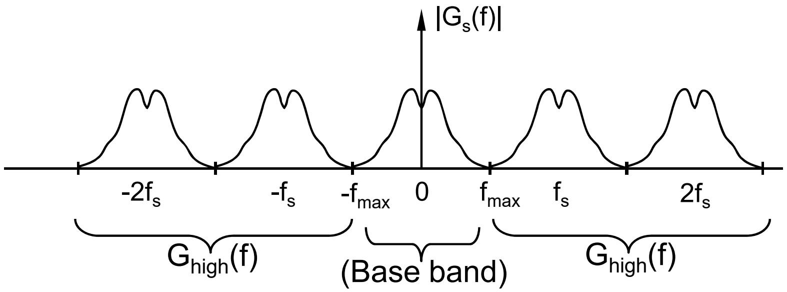
Filtered image

Insight: Analysis in Frequency Domain



Exploit Well-Known Properties of Fourier Transforms

1. Multiplication in spatial domain $\leftrightarrow$ convolution in frequency domain.
2. Fourier transform of an impulse train is itself an impulse train.
3. $G_s(f)$ is the original spectrum $G(f)$ replicated in the frequency domain with period f_s.



$G_s(f) = G(f) + G_{high}(f)$ $\rightarrow$ introduced by sampling

Solution to Reconstruction

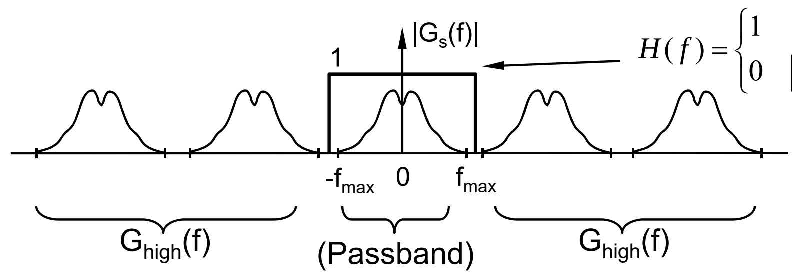
Discard replicas $G_{high}(f)$, leaving baseband $G(f)$ intact.

Two Provisions for exact reconstruction:

- The signal must be bandlimited. Otherwise, the replicas would overlap with the baseband spectrum.
- $f_s > 2f_{max}$ (Nyquist frequency)

Ideal Filter in the Frequency Domain: Box Filter

- Assuming provisions hold, we can discard G_{high} by multiplying $G_s(f)$ with a box filter in the frequency domain.

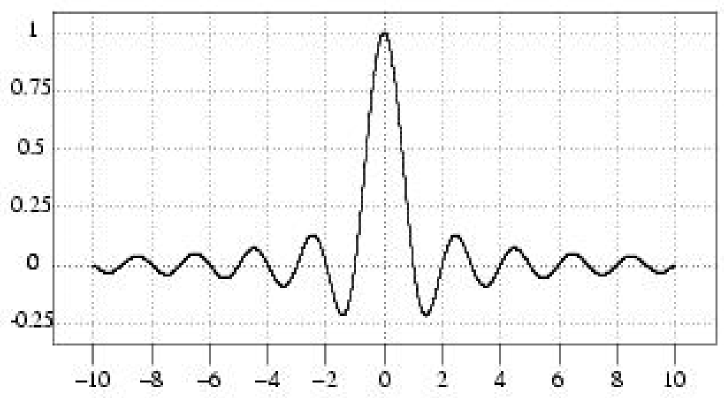


$$H(f) = \begin{cases} 1 & |f| < f_{max} \quad \text{(passband)} \\ 0 & |f| \geq f_{max} \quad \text{(stopband)} \end{cases}$$

Ideal Filter in the Spatial Domain: Sinc Function

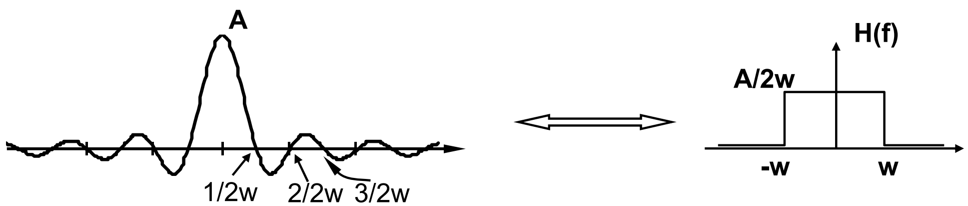
The ideal lowpass filter in the spatial domain is the inverse Fourier transform of the ideal box:

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Reciprocal Relationship

- Reciprocal relationship between spatial and frequency domains:



- For interpolation, A must equal 1 (unity gain when sinc is centered at samples; pass through 0 at all other samples)

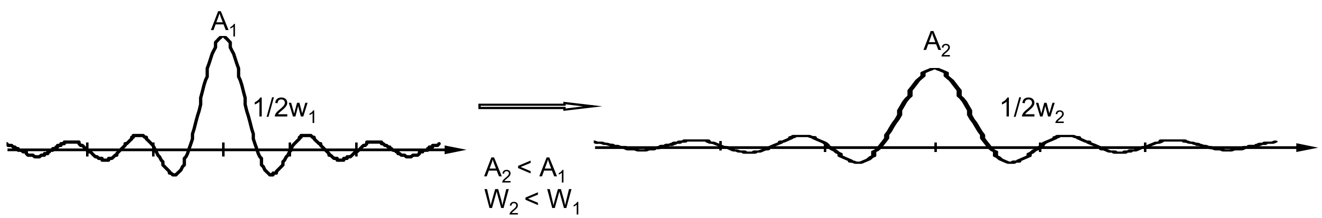
$W = f_{max} = 0.5$ cycle / pixel for all digital images

Highest freq: on-off sequence

1 cycle is 2 pixels (black, white, black, white), therefore ½ cycle per pixel is f_{max}.

Blurring

- To blur an image, W↓ and A↓ so that A/2W remains at 1.



- To interpolate among sparser samples, A=1 and W↓ which means that (A/2W)↑. Since sampling decreases the amplitude of the spectrum, (A/2W) ↑ serves to restore it.

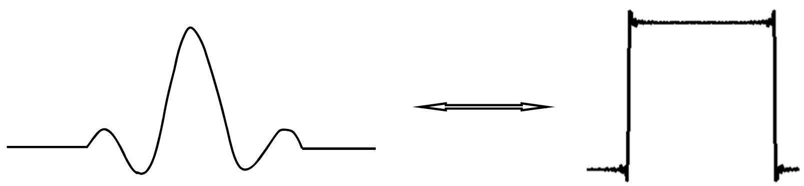
$$g(x) = \text{sinc}(x) * g_s(x)$$
$$= \int_{-\infty}^{\infty} \text{sinc}(\lambda) g_s(x-\lambda) d\lambda$$

Note : sinc requires infinite number of neighbours : impossible.

- Ideal low-pass filtering is impossible in the spatial domain. It is not impossible in the frequency domain for finite data streams.

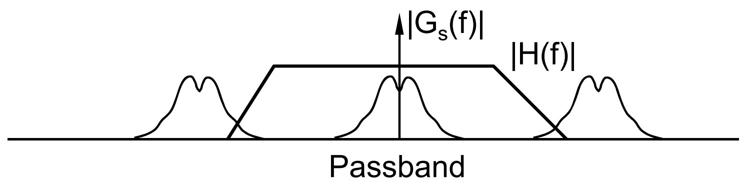
Nonideal Reconstruction

- Possible solution to reconstruction in the spatial domain: truncated sinc.



Ripples;
Gibbs phenomenon

- Alternate solution: nonideal reconstruction



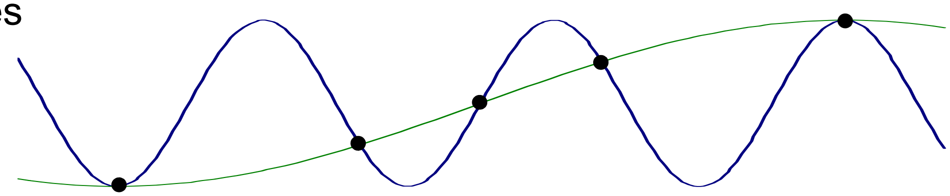
- Nonideal because it attenuates the higher frequencies in the passband and doesn't fully suppress the stopband.

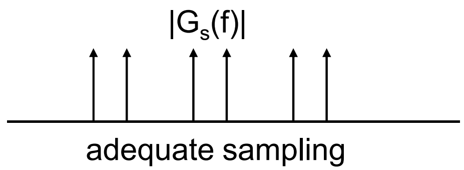
Aliasing

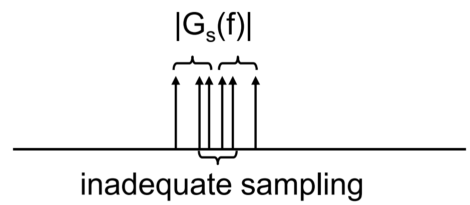
If $f_s < 2f_{max}$ our signal is undersampled.

If f_{max} is infinite, our signal is not bandlimited.

Either way, we have aliasing: high frequencies masquerade, or alias as, low frequencies







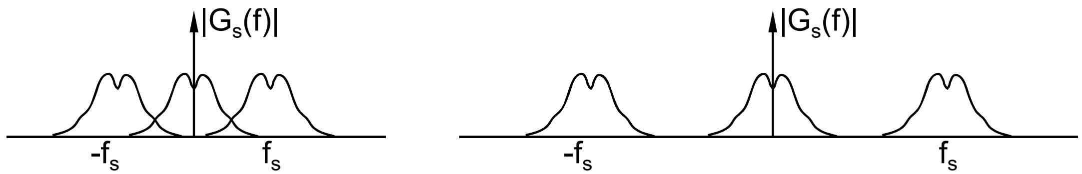
(overlapping spectrum replicas)

Temporal aliasing: stagecoach wheels go backwards in films (sampled at 24 frames/sec).

Antialiasing

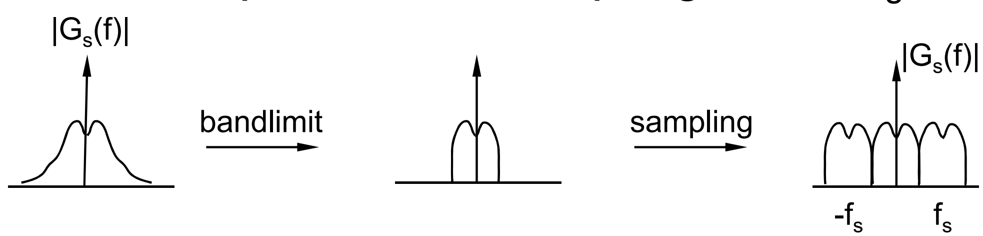
Two approaches to antialiasing for combating aliasing:

1. Sample input at higher rate: increase f_s



low f_s (overlaps) higher f_s (no overlaps)

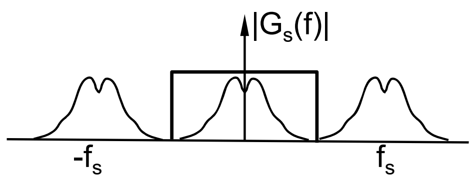
2. bandlimit input before sampling at low f_s



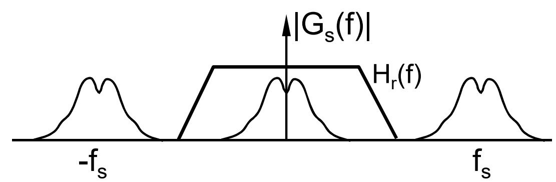
Antialiasing

Intuition for increasing f_s:

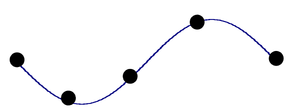
Higher sampling rates permit sloppier reconstruction filters.



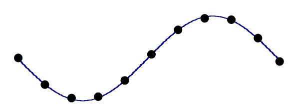
Marginally acceptable f_s
Requires ideal lowpass filter



high f_s allows nonideal reconstruction filter.
This adequate because $H_r(f)$ doesn't taper off until $|F|>|F_{max}|$

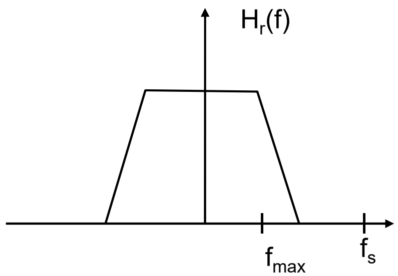


Pushed to limits; use sinc



Many samples: linear interpolation adequate

Nonideal Reconstruction Filter



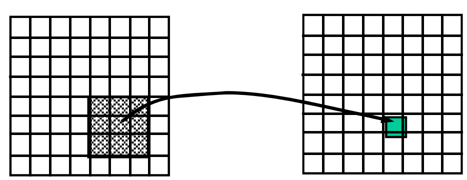
$$H_r(f) = \begin{cases} 1 & 0 \le |f| \le f_{\max} \\ \ne 0 & f_{\max} < |f| < f_s - f_{\max} \\ 0 & |f| \ge f_s - f_{\max} \end{cases}$$

Nonideal Reconstruction Filter

Intuition for bandlimiting input:

Inability to change fs: e.g., restricted by display resolution.

For example, an image in perspective may require infinite pixels to capture all visual details. However, only finite # pixels available, e.g. 512 x 512



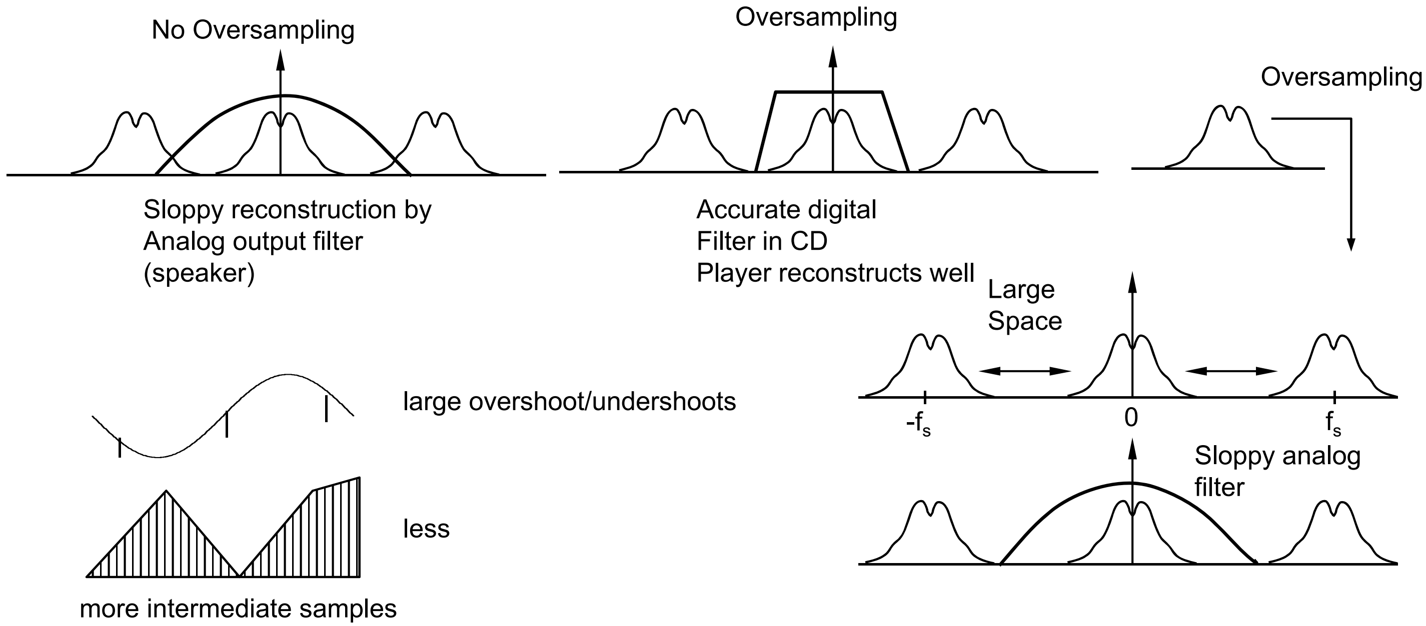
Many pixels in oblique image map to one output pixel. Must blur neighborhood before returning a single value to the output image. In general, dull signals don't need bandlimiting, rich ones do.

Example: Oversampling CD players

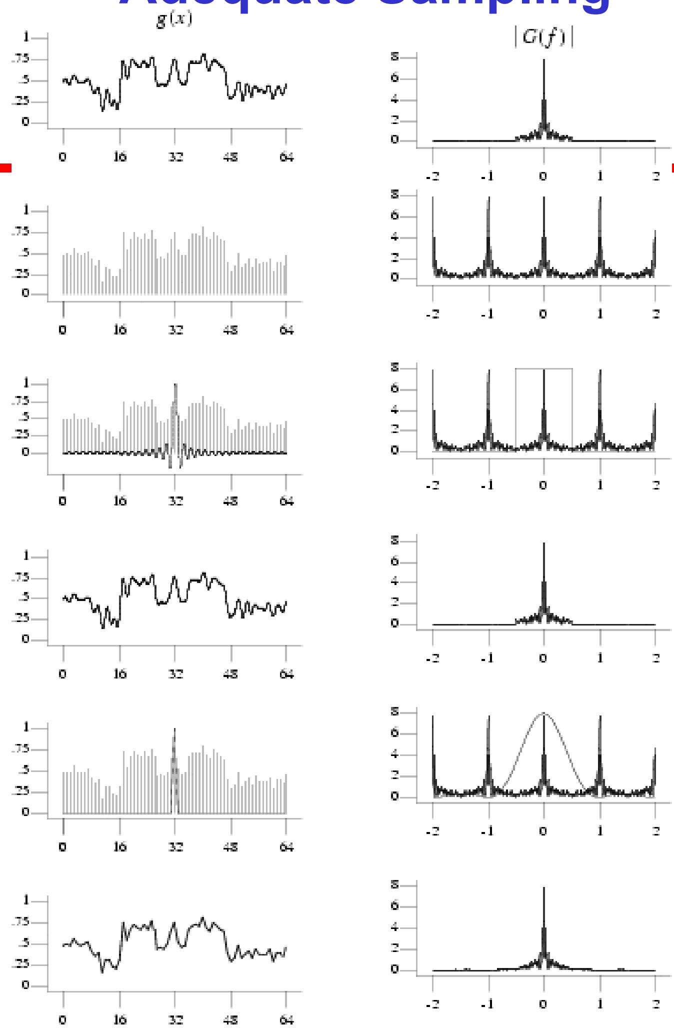
f_s = 44.1 kHz (sample/second)

= 20 kHz * 2 + 4.1 kHz = 44.1KHz

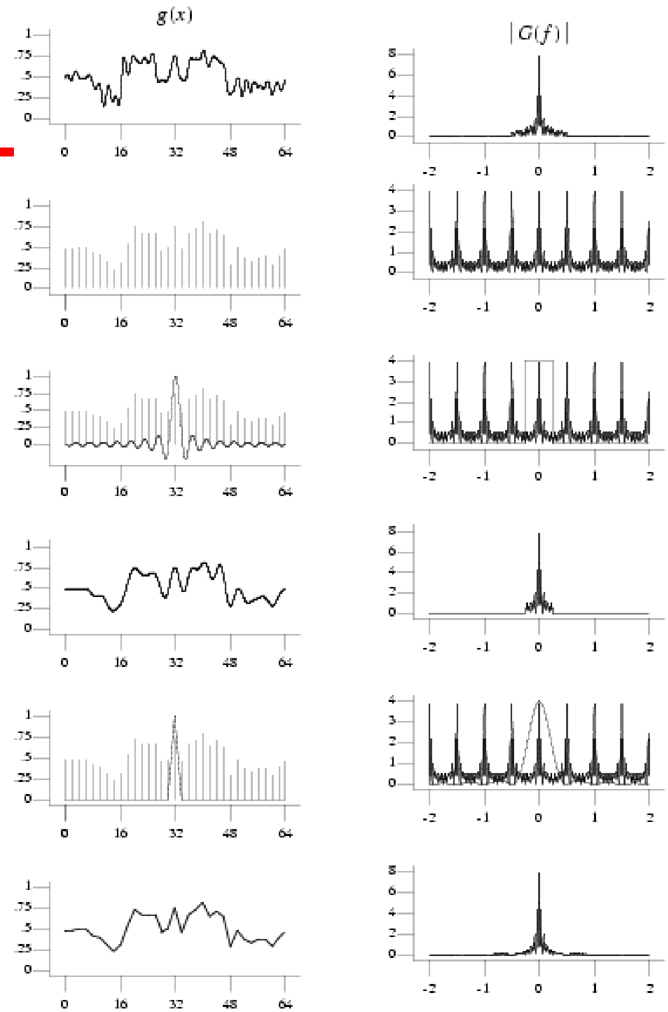
f_{max} — *Nyq rate* — *cushion*



Adequate Sampling



Inadequate Sampling



Antialiasing